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The space $C_p(X)$ admits a dense exponentially separable subspace when X is metrizable ☆

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We prove that $C_p(X)$ admits a dense exponentially separable for any metrizable space X and, answering a question in [16], we give an example of a pseudocompact ω-monolithic space such that $C_p(X)$ does not admit dense functionally countable subspaces. In a similar sense, solving consistently a problem in [11], we prove that $C_p(X)$ may not always contain a dense subspace of countable functional tightness. In other direction, and answering a question posed in [6], we characterize compact spaces for which their Alexandroff doubles have a Lindelöf C_p; we also give a short proof of a result in [19] about a consistent characterization of the Lindelöf property in C_p-spaces over Hattori spaces.

1. Introduction

This paper contains some results on function spaces $C_p(X)$, the first part is dedicated to understanding when, given a property, the space $C_p(X)$ admits a dense subspace having the given property, and the second part focuses on some results concerning the Lindelöf property in spaces $C_p(X)$ when the base space X is obtained from some specific constructions.

Functional countability arose a long time ago in the study of function spaces over certain spaces. Exponential separability is an interesting strengthening of functional countability introduced recently in [15]. In the paper [16], it was proved that the space $\mathbb{R}^\kappa$ admits a dense exponentially separable subspace for any cardinal κ. We extend this result by showing that the space $C_p(X)$ admits a dense exponentially separable subspace for any metrizable space X. In the paper [17], it was proved that, for a zero-dimensional countably compact space X, the space $C_p(X)$ admits a dense functionally countable subspace if and only if X is ω-monolithic. Solving Question 4.3 in [17] in the negative, we show that there exists a pseudocompact

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ω-monolithic space X such that $C_p(X)$ does not admit dense functionally countable subspaces. Addressing another question in [17], we also prove that $C_p(X)$ contains a dense functionally countable subspace when X is a P-space of cardinality at most ω_1.

Functional tightness is a property that arises naturally in the context of function spaces. It is known that functional tightness is the dual property of realcompactness, in the sense that $C_p(X)$ is functionally countable if and only if X is realcompact. We solve Problem 4.3.5 in [11], under the Continuum Hypothesis, by proving that there exists a space X such that $C_p(X)$ does not admit a dense subspace of countable functional tightness.

On the other hand, the study of the Lindelöf property in function spaces is one of the most interesting and productive topics of research in this area. Buzyakova studied in [6] the preservation of the Lindelöf property in function spaces $C_p(X)$ under slight modifications in the base space X. We answer Question 3.15 in [6] by providing a characterization of compact spaces X for which their Alexandroff doubles have a Lindelöf C_p. On the other hand, in the last years Hattori spaces have attracted the attention of topologists in many senses, and their function spaces are not the exception. The paper [19] presents a consistent characterization of the Lindelöf property in C_p-spaces over Hattori spaces; we provide a short proof of the above characterization, relaxing the corresponding consistency requirements. In addition, we give a characterization of those Hattori spaces that have Lindelöf square.

In this paper, we use the following notation. All spaces are assumed to be Tychonoff. By $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the sets of integer, rational, and real numbers, respectively, with the usual topology. The unitary interval $[0,1]$ in $\mathbb{R}$ is denoted by I and $2 = \{0,1\}$. The symbol κ denotes a cardinal number, ω is the first infinite cardinal number, ω_1 is the first uncountable cardinal number and $\mathfrak{c}$ is the continuum. Given a subset $X \subset \mathbb{R}$ and cardinals κ and λ, the Σ_λ-*product* of X^κ is the subspace

$$\Sigma_\lambda X^\kappa = \{x \in X^\kappa : |\{\alpha \in \kappa : x(\alpha) \neq 0\}| < \lambda\}$$

of the product X^κ. Besides, $\Sigma X^\kappa = \Sigma_{\omega_1} X^\kappa$. Given a cardinal κ, the one-point compactification of the discrete space of cardinality κ is denoted as $A(\kappa)$. By βX we denote the Stone-Čech compactification of a space X.

The symbol $C(X,Y)$ denotes the set of all continuous functions from a space X to a space Y. The space $C_p(X,Y)$ is the set $C(X,Y)$ endowed with the topology of pointwise convergence. We take $C(X) = C(X,\mathbb{R})$ and $C_p(X) = C_p(X,\mathbb{R})$. A set $A \subset C_p(X)$ is an *algebra* if it is closed under sums and products. Given $A \subset C_p(X)$, the $\mathbb{Q}$-*algebra* generated by A is the minimal algebra of $C_p(X)$ which contains A and all constant functions in $C_p(X,\mathbb{Q})$ (see [17]). Let Y be a subspace of a space X, define the *restriction function* $\pi_Y : C_p(X) \to C_p(Y)$ by $\pi_Y(f) = f \restriction_Y$ for each $f \in C_p(X)$. Given a continuous function $f : X \to Y$, define its *dual function* $f^* : C_p(Y) \to C_p(X)$ by $f^*(g) = g \circ f$ for any $g \in C_p(Y)$. For a subspace $A \subset X$, an *extender* is a function $\varphi : C_p(A) \to C_p(X)$ such that $\pi_A(\varphi(f)) = f$ for each $f \in C_p(A)$.

The closure of a subset A of a space X is denoted as $\mathrm{cl}(A)$. A *network* $\mathcal{N}$ for X is a family of subsets of X such that each open set in X is the union of a subfamily of $\mathcal{N}$. A space with a countable network is called *cosmic*. A space X is ω-*monolithic* if $\mathrm{cl}(A)$ is cosmic for every countable set $A \subset X$. A space X is K-*analytic* if there exists a space Y that admits a perfect function onto the space of irrational numbers ω^ω and a continuous function onto X. A space X is *Lindelöf* Σ if there exists a space Y that admits a perfect function onto a second countable space M and a continuous function onto X. The rest of the terminology and notation is standard and follows [4].

2. Exponentially separable dense subspaces of $C_p(X)$

Let $\mathcal{P}$ be a topological property such that: if $C_p(X)$ has $\mathcal{P}$, then $C_p(Y)$ has $\mathcal{P}$ for any subspace $Y \subset X$; and such that, if $X = \prod\{X_n : n < \omega\}$ and $C_p(X_n)$ has $\mathcal{P}$ for each $n < \omega$, then $C_p(X)$ has $\mathcal{P}$. If we wish to verify that, for a given metric space X, $C_p(X)$ satisfies a topological property $\mathcal{P}$, by the Kowalsky's Theorem, it is sufficient to verify that $C_p(Y)$ satisfies the property $\mathcal{P}$, where Y is the hedgehog space (see [4, Example 4.1.15]). For example, Arhangel'skii proved that for any metrizable space X, $C_p(X)$ contains a compact subset which separates the points of X (see [2, Theorem IV 1.22]).

In this section we use a similar method to prove that $C_p(X)$ admits a dense exponentially separable subspace for any metrizable space X. Let us recall the definition of exponential separability and prove some auxiliary results.

Given a family $\mathcal{F}$ of subsets of X, say that a set $A \subset X$ is *strongly dense* in $\mathcal{F}$ if $A \cap \bigcap \mathcal{G} \neq \emptyset$ whenever $\mathcal{G} \subset \mathcal{F}$ and $\bigcap \mathcal{G} \neq \emptyset$. A space X is called *exponentially separable* if for every countable family $\mathcal{F}$ of closed sets there is a countable subset A of X that is strongly dense in $\mathcal{F}$.

Proposition 2.1. *Assume that $\mathcal{P}$ is a topological property preserved under continuous functions. If $Y \subset X$ and the space $C_p(X)$ admits a dense subspace with the property $\mathcal{P}$, then $C_p(Y)$ also admits a dense subspace with the property $\mathcal{P}$.*

Proof. Suppose that A is a dense in $C_p(X)$ and has the property $\mathcal{P}$. Consider the restriction function $\pi_Y : C_p(X) \to C_p(Y)$. Note that $\pi_Y(C_p(X))$ is dense in $C_p(Y)$ (see Problem 152 (i) of [11]) and, in particular, $B = \pi_Y(A)$ is dense in $C_p(Y)$. Since $\mathcal{P}$ is preserved under continuous images, the space B also has the property $\mathcal{P}$. □

Proposition 2.2. *If $A \subset C_p(X)$ is a $\mathbb{Q}$-algebra which separates the points of X, then A is dense in $C_p(X)$.*

Proof. For each $f \in A$ let $\hat{f} \in C_p(\beta X)$ be the continuous extension of f to the Stone-Čech compactification βX of X. Define an equivalence relation in βX as follows: $x \sim y$ if $\hat{f}(x) = \hat{f}(y)$ for each $f \in A$. Denote by K and q the quotient space and the quotient function obtained from this relation. Observe that since the $\mathbb{Q}$-algebra A separates the points of X, the function $p = q\restriction_X : X \to Y := q(X)$ is a condensation. For each $f \in A$ let $\check{f} \in C_p(K)$ be the continuous function induced by $\hat{f}$ on the quotient space K satisfying the equality $\check{f} \circ q = \hat{f}$. Observe that both $\hat{A} = \{\hat{f} : f \in A\} \subset C_p(\beta X)$ and $\check{A} = \{\check{f} : f \in A\} \subset C_p(K)$ are $\mathbb{Q}$-algebras of continuous functions. It follows from the construction that $\check{A}$ separates the points of K, so the Stone-Weierstrass Theorem implies that $\check{A}$ is uniformly dense in $C_p(K)$. It follows that $\check{A}\restriction_Y = \{\check{f}\restriction_Y : f \in A\}$ is dense in $C_p(Y)$. Finally, since p is a condensation, the space $A = p^*(\check{A}\restriction_Y)$ is dense in $C_p(X)$. □

Theorem 2.3. *If X is a metrizable space, then $C_p(X)$ contains a dense exponentially separable subspace.*

Proof. Let κ be the cardinality of X and consider the hedgehog space J with κ spines: Consider an equivalence relation on $I \times \kappa$ defined as $\langle t, \alpha\rangle \sim \langle s, \beta\rangle$ if $s = t$ and $\alpha = \beta$ or $s = 0 = t$. Then J is set of equivalence classes induced on $I \times \kappa$ by this relation and the metric on J is defined as follows.

$$d([\langle t, \alpha\rangle], [\langle s, \beta\rangle]) = \begin{cases} |s - t| & \text{if } \alpha = \beta; \\ s + t & \text{if } \alpha \neq \beta. \end{cases}$$

We claim that the space $C_p(J)$ admits a dense exponentially separable subspace that separates the points of J. Indeed, for each $\alpha < \kappa$ let $f_\alpha \in C_p(J)$ be the function defined as

$$f_\alpha([\langle t, \beta\rangle]) = \begin{cases} t & \text{if } \alpha = \beta; \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Then the space $K = \{f_\alpha : \alpha < \kappa\} \cup \{\zeta\}$, where $\zeta \in C_p(J)$ is the zero constant function, is a compact subspace of $C_p(J)$ homeomorphic to the one-point compactification of the discrete space of cardinality κ.

Observe that K is a compact scattered space and that, by Corollary 3.14 in [15], it is exponentially separable. Furthermore, the space K separates the points of J.

Now consider the space $M = J^\omega$. We will prove that $C_p(M)$ also admits an exponentially separable subspace that separates the points of M. For each $n < \omega$ let $p_n : M \to J$ be the canonical projection onto the n-th coordinate and let $p_n^* : C_p(J) \to C_p(M)$ be the dual function of p_n. Since the function p_n^* is a homeomorphism, $K_n = p_n^*(K)$ is homeomorphic to K and hence is a scattered compact space. For each $n \leq \omega$ let $S_n = \bigcup\{K_k : k \leq n\} \subset C_p(M)$. For each $n < \omega$, the space S_n is the union of finitely many scattered compact spaces and hence is a scattered compact space. In particular, the space S_n exponentially separable. The space $S = S_\omega$ is a countable union of exponentially separable spaces, so we can apply Proposition 3.2 (d) of [15], to see that S is exponentially separable. Furthermore, it is easy to verify that S separates the points of M.

Now, we will prove that the space $C_p(M)$ admits a dense exponentially separable subspace. Let A be the $\mathbb{Q}$-algebra in $C_p(M)$ generated by S. Observe that $A = \bigcup\{A_n : n < \omega\}$, where each A_n is the continuous image of $T_n = S_{m_n}^{k_n} \times \mathbb{Q}^{k_n}$ for some $m_n, k_n < \omega$. Since S_{m_n} is compact and scattered, the space $S_{m_n}^{k_n}$ is compact and scattered, and consequently it is exponentially separable. Since T_n is the union of countably many copies of $S_{m_n}^{k_n}$ and exponential separability is preserved under countable unions, the space T_n is exponentially separable. Moreover, by Proposition 3.2 (c) in [15] exponential separability is also preserved under continuous images, so the space A_n is exponentially separable. The space A, being a countable union of exponentially separable spaces, is exponentially separable. Finally, Proposition 2.2 implies that A is dense in $C_p(M)$.

Finally, by the Kowalski's Theorem (see [4, Theorem 4.4.9]), the space X can be embedded in M. Then we can apply the fact that exponential separability is preserved under continuous images and Proposition 2.1 to see that $C_p(X)$ admits a dense exponentially separable subspace. □

3. Functionally countable dense subspaces of $C_p(X)$

Shakhmatov constructed, for any cardinal $\kappa \geq \mathfrak{c}$, a pseudocompact dense subspace $X \subset I^\kappa$ such that any countable subset of X is closed, discrete and C-embedded in X (see [2, Example I.2.5]). We will use a variant of the space constructed by Shakhmatov to provide an example of a pseudocompact ω-monolithic space X for which $C_p(X)$ does not admit any functionally countable dense subspace.

Let us recall the definition of functional countability and a very useful result; the Arhangel'skii Factorization Theorem (see Corollary 1.7.8 in [3]).

A space X is *functionally countable* if $f(X)$ is countable for any continuous function $f : X \to \mathbb{R}$.

Theorem 3.1. *Let $X = \prod\{X_\alpha : \alpha \in \kappa\}$ be a product of cosmic spaces and let Y be a dense subspace of X. Then every continuous function $f : Y \to \mathbb{R}$ can be factorized through the projection $p_D : X \to \prod\{X_\alpha : \alpha \in D\}$ for some countable set $D \subset \kappa$, in the sense that there exists a continuous function $g : p_D(Y) \to \mathbb{R}$ such that $f = g \circ p_D \restriction_Y$.*

Theorem 3.2. *There exists a pseudocompact ω-monolithic space X such that $C_p(X)$ does not admit dense functionally countable subspaces.*

Proof. First, we construct the corresponding space X. Let $\kappa = 2^{\omega_1}$, let $\mathcal{D}$ be the family of all subsets of κ of cardinality at most ω_1 and let $Z = \Sigma_{\omega_2} I^\kappa$. Observe that $\mathcal{D}$ and Z have cardinality κ. Choose an enumeration $\{D_\alpha : \alpha < \kappa\}$ of $\mathcal{D}$ where each element admits κ-many repetitions and, similarly, choose an enumeration $\{z_\alpha : \alpha < \kappa\}$ of Z where each element appears κ-many times. For each $\alpha < \kappa$ define $x_\alpha \in I^\kappa$ as follows:

$$x_\alpha(\gamma) = \begin{cases} z_\alpha(\gamma) & \text{if } \gamma \le \alpha; \\ 1 & \text{if } \gamma > \alpha \text{ and } \alpha \in D_\gamma; \\ 0 & \text{if } \gamma > \alpha \text{ and } \alpha \notin D_\gamma. \end{cases}$$

Then we consider the subspace $X = \{x_\alpha : \alpha < \kappa\} \subset I^\kappa$. For each set $D \subset \kappa$ let $p_D : I^\kappa \to I^D$ be the natural projection. Then X has the following properties:

(1) For each $D \in \mathcal{D}$ we have the equality $p_D(X) = I^D$.
(2) Every set $B \subset X$ of cardinality at most ω_1 is discrete and closed in X.
(3) Any $f \in C_p(X)$ factorizes through p_D for some countable set $D \subset \kappa$.

To verify (1), given $D \in \mathcal{D}$ and $y \in I^D$, we can choose $z \in Z$ such that $p_D(z) = y$. If $\alpha < \kappa$ is an ordinal such that $D \subset \alpha$ and $z_\alpha = z$, then observe that $p_D(x_\alpha) = p_D(z_\alpha) = p_D(z) = y$. To see that (2) holds, pick a subset $B = \{x_{\alpha_\beta} : \beta < \omega_1\}$ of X. Given $\delta < \omega_1$, let $D = \{\alpha_\delta\} \in \mathcal{D}$. Since the cofinality of κ is greater than ω_1, we can choose $\gamma < \kappa$ such that $D_\gamma = D$ and $\alpha_\beta < \gamma$ for each $\beta < \omega_1$. Let $U = \{x \in X : x(\gamma) = 1\}$ and observe that, given $\beta < \omega_1$, we have $x_{\alpha_\beta} \in U$ if and only if $\alpha_\beta \in D_\gamma = D = \{\alpha_\delta\}$, and so $U \cap D = \{x_{\alpha_\delta}\}$. This shows that B is discrete and, since B is arbitrary, we conclude that B is also closed in X. Condition (1) implies that X is dense in I^κ, so condition (3) follows directly from Theorem 3.1.

Observe that condition (1) implies that the function in condition (3) is bounded, so X is pseudocompact. In addition, condition (2) implies that X is ω-monolithic. This concludes the construction of the space X.

Now, we will show that $C_p(X)$ does not admit dense functionally countable subspaces. To this end, fix a dense subspace A of $C_p(X)$. We will use the following notation. Given $x \in X$, consider the function $\varphi_x : C_p(X) \to \mathbb{R}$ defined by $\varphi_x(f) = f(x)$ for each $f \in C_p(X)$. Observe that φ_x is the restriction to $C_p(X)$ of the projection from $\mathbb{R}^X$ onto the x-coordinate and hence it is continuous. In order to show that A is not functionally countable, we shall prove that there exists $x \in X$ such that $\varphi_x(A)$ is uncountable. For the construction of x we will define recursively sequences $\{y_\alpha : \alpha < \omega_1\} \subset X$, $\{f_\alpha : \alpha < \omega_1\} \subset A$ and $\{D_\alpha : \alpha < \omega_1\} \subset \mathcal{D}$ as follows.

Choose $f_0 \in A$ and $y_0 \in X$ arbitrarily. By (3) there exists a countable set $D_0 \subset \kappa$ such that f_0 factorizes through p_{D_0}. Assume that for some ordinal $\alpha < \omega_1$ we have constructed sequences $\{y_\beta : \beta < \alpha\} \subset X$, $\{f_\beta : \beta < \alpha\} \subset A$ and $\{D_\beta : \beta < \alpha\} \subset \mathcal{D}$ such that, for each $\gamma < \beta < \alpha$, the following conditions hold:

(4) $f_\beta(y_\beta) \ne f_\gamma(y_\gamma)$;
(5) f_β factorizes through p_{D_β};
(6) $D_\gamma \subset D_\beta$ and $p_{D_\gamma}(y_\beta) = p_{D_\gamma}(y_\gamma)$.

Put $D'_\alpha = \bigcup\{D_\beta : \beta < \alpha\} \in \mathcal{D}$ and note that condition (6) implies that $w_\alpha = \bigcup\{p_{D_\beta}(y_\beta) : \beta < \alpha\}$ is an element of $I^{D'_\alpha}$. Let $q_\alpha = p_{D'_\alpha} \restriction_X$. Choose $\delta \in \kappa \setminus D'_\alpha$ and fix two points $y'_\alpha, y''_\alpha \in X$ such that $y'_\alpha(\delta) = 0$, $y''_\alpha(\delta) = 1$ and $q_\alpha(y'_\alpha) = w_\alpha = q_\alpha(y''_\alpha)$. Since A is a dense subset of $C_p(X)$, there exists $f_\alpha \in A$ such that $f_\alpha(y'_\alpha) \ne f_\alpha(y''_\alpha)$. By (3) there exists a set $D_\alpha \in \mathcal{D}$ and a continuous function $g_\alpha : I^{D_\alpha} \to \mathbb{R}$ such that $f_\alpha = g_\alpha \circ p_{D_\alpha} \restriction_X$. We can assume that $D'_\alpha \subset D_\alpha$. Observe that if $r_\alpha : I^{D_\alpha} \to I^{D'_\alpha}$ is the canonical projection, then (1) implies that $f_\alpha(q_\alpha^{-1}(w_\alpha)) = g_\alpha(p_{D'_\alpha}(q_\alpha^{-1}(w_\alpha))) = g_\alpha(r_\alpha^{-1}(w_\alpha))$, being the continuous image of the connected space $r_\alpha^{-1}(w_\alpha)$, is connected. Since $f_\alpha(y'_\alpha), f_\alpha(y''_\alpha) \in f_\alpha(q_\alpha^{-1}(w_\alpha))$, we deduce that $f_\alpha(q_\alpha^{-1}(w_\alpha))$ has cardinality $\mathfrak{c}$. Choose $y_\alpha \in q_\alpha^{-1}(w_\alpha)$ such that $f_\alpha(y_\alpha) \notin \{f_\beta(y_\beta) : \beta < \alpha\}$. Then conditions (4)-(6) are satisfied by $\{y_\beta : \beta < \alpha + 1\}$, $\{f_\beta : \beta < \alpha + 1\}$ and $\{D_\beta : \beta < \alpha + 1\}$.

When the construction finishes, we obtain sequences $\{y_\alpha : \alpha < \omega_1\} \subset X$, $\{f_\alpha : \alpha < \omega_1\} \subset A$ and $\{D_\alpha : \alpha < \omega_1\} \subset \mathcal{D}$ satisfying conditions (4)-(6). In order to construct the required point $x \in X$, choose $D = \bigcup\{D_\alpha : \alpha < \omega_1\} \in \mathcal{D}$, observe that condition (6) implies that $w = \bigcup\{p_{D_\alpha}(y_\alpha) : \alpha < \omega_1\}$ is an element of I^D and note that condition (1) implies that there exists $x \in X$ satisfying $p_D(x) = w$ and consequently such that $p_{D_\alpha}(x) = p_{D_\alpha}(w) = p_{D_\alpha}(x_\alpha)$ for each $\alpha < \omega_1$. To verify that x satisfies the required properties, observe that condition (5) implies that $f_\alpha(X) = f_\alpha(x_\alpha)$ for each $\alpha < \omega_1$, so condition (4) implies that $\{f_\alpha(X) : \alpha < \omega_1\}$ is uncountable; besides $\{f_\alpha(X) : \alpha < \omega_1\} = \{\varphi_x(f_\alpha) : \alpha < \omega_1\} \subset \varphi_x(A)$ and so $\varphi_x(A)$ is uncountable. Therefore, A is not functionally countable. □

The following result shows that, at least consistently, the space obtained in the proof of Theorem 3.2 cannot be constructed using ω instead of ω_1.

Theorem 3.3. *If X has cardinality ω_1 and all countable subsets of X are closed and discrete, then $C_p(X)$ admits a dense functionally countable subspace.*

Proof. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be an enumeration of X and choose $X_\alpha = \{x_\beta : \beta < \alpha\}$ for each $\alpha < \omega_1$. We will construct recursively a family $\{A_\alpha : \alpha < \omega_1\}$ of countable $\mathbb{Q}$-algebras in $C_p(X)$ as follows.

Let A_0 be the $\mathbb{Q}$-algebra generated by the zero constant function ζ in $C_p(X)$. Given $\alpha < \omega_1$, assume that for each $\beta < \alpha$ we have defined a countable $\mathbb{Q}$-algebra $A_\beta \subset C_p(X)$ satisfying, for each $\gamma < \beta$, the following properties:

(1) $\pi_{X_\beta}(A_\beta)$ is dense in $C_p(X_\beta)$;
(2) $\pi_{X_\gamma}(A_\beta) = \pi_{X_\gamma}(A_\gamma)$;
(3) $A_\gamma \subset A_\beta$.

If $\alpha = \beta + 1$, then choose $f_\alpha \in C_p(X)$ such that $f_\alpha(X_\beta) \subset \{0\}$ and $f_\alpha(x_\alpha) = 1$ and let $A_\alpha \subset C_p(X)$ be the countable $\mathbb{Q}$-algebra generated by $A_\beta \cup \{f_\alpha\}$. Then A_α separates the points of X_α and, by applying Proposition 2.2, we deduce that $\pi_{X_\alpha}(A_\alpha)$ is dense in $C_p(X_\alpha)$. If α is a limit ordinal, consider the countable $\mathbb{Q}$-algebra $A_\alpha = \bigcup\{A_\beta : \beta < \alpha\} \subset C_p(X)$. Observe that A_α separates the points of X_α and, as before, we deduce that $\pi_{X_\alpha}(A_\alpha)$ is dense in $C_p(X_\alpha)$. In both cases, conditions (1)-(3) are satisfied for each $\gamma < \beta < \alpha+1$.

Once the construction is finished, consider the $\mathbb{Q}$-algebra $A = \bigcup\{A_\alpha : \alpha < \omega_1\}$. Observe that A separates the points of X and so we can apply Proposition 2.2 to see that A is a dense subset of $C_p(X)$. Condition (2) of the construction implies that $\pi_{X_\alpha}(A) = \pi_{X_\alpha}(A_\alpha)$ is countable for each $\alpha < \omega_1$. Finally, to see that A is functionally countable, choose a continuous function $\varphi : A \to \mathbb{R}$. By Theorem 3.1 we can find a countable set $B \subset X$ such that φ factorizes through $\pi_B \restriction_A$. Choose $\alpha < \omega_1$ such that $B \subset X_\alpha$; then φ factorizes through $\pi_{X_\alpha} \restriction_A$, that is, there exists a continuous function $\phi : \pi_{X_\alpha}(A) \to \mathbb{R}$ such that $\varphi = \phi \circ \pi_{X_\alpha} \restriction_A$. So, we conclude that $\varphi(A) = \phi(\pi_{X_\alpha}(A))$ is countable. □

Corollary 3.4. *If X is a P-space of cardinality at most ω_1, then $C_p(X)$ admits a dense functionally countable subspace.*

4. Dense subspaces of $C_p(X)$ with countable functional tightness

It was proved in [1] that, under the continuum hypotheses, the space $C_p(\Sigma 2^{\omega_1})$ does not admit dense subspaces having countable tightness. Here we improve this result by showing that, under the same assumption, the space $C_p(\Sigma 2^{\omega_1})$ does not admit dense subspaces having countable functional tightness.

Recall that a space X has *countable functional tightness* if every function $f : X \to \mathbb{R}$ satisfying that $f \restriction_D$ is continuous for every countable set $D \subset X$ must be continuous.

Theorem 4.1. *Under CH, the space $C_p(\Sigma 2^{\omega_1})$ does not contain dense subspaces having countable functional tightness.*

Proof. Let $K = 2^{\omega_1}$ and $X = \Sigma 2^{\omega_1}$. For each $\alpha < \omega_1$ consider the natural projection $p_\alpha : K \to 2^\alpha$. Given $f \in C_p(X)$, by Theorem 3.1, we can fix the least ordinal $\sigma_f < \omega_1$ such that f factorizes through p_{σ_f}. Choose a continuous function $g : 2^{\sigma_f} \to \mathbb{R}$ such that $f = g \circ p_{\sigma_f} \restriction_X$; observe that this equality implies that f is bounded and that $\hat{f} := g \circ p_{\sigma_f} \in C_p(K)$ is a continuous extension of f. It follows that X is pseudocompact and C-embedded in K. Since K is compact and X is dense in K, we deduce that $K = \beta X$ is the Stone-Čech compactification of X. Furthermore, the restriction function $\pi : C_p(K) \to C_p(X)$ is a continuous bijection. Since X is G_δ-dense and C-embedded in K, we can apply Proposition 4.11 in [9] to see that $\pi \restriction_B : B \to \pi(B)$ is a homeomorphism for every countable set $B \subset C_p(K)$.

Given $x \in K$, let $\varphi_x : C_p(K) \to \mathbb{R}$ be the evaluation function defined as $\varphi_x(g) = g(x)$ for each $g \in C_p(K)$. Since φ_x is simply a projection, φ_x is a continuous function. We define $\psi_x : C_p(X) \to \mathbb{R}$ by the rule $\psi_x(f) = \varphi_x(\pi^{-1}(f)) = \varphi_x(\hat{f})$ for each $f \in C_p(X)$. Then the function ψ_x is not necessarily continuous, but since φ_x is continuous and $\pi \restriction_B : B \to \pi(B)$ is a homeomorphism for every countable set $B \subset C_p(K)$, we deduce that $\psi_x \restriction_C$ is continuous for every countable set $C \subset C_p(X)$. To finish the proof, given an arbitrary dense subset $A \subset C_p(X)$, we will construct recursively a point $x \in K$ such that the corresponding function $\psi_x \restriction_A$ is not continuous, proving so that A does not have countable functional tightness.

Fix an arbitrary dense subspace $A \subset C_p(X)$. By the homogeneity of $C_p(X)$, without loss of generality, we can assume that the zero constant function ζ on X is an element of A. Given $x \in X$, let $\sigma_x < \omega_1$ be the least ordinal such that $\operatorname{supp}(x) := \{\alpha < \omega_1 : x(\alpha) \neq 0\} \subset \sigma_x$. Denote by $\mathcal{F}$ the family of all non-empty finite subsets of X. Note that under the Continuum Hypothesis both X and $\mathcal{F}$ have cardinality ω_1. Choose an enumeration $\{(F_\alpha, n_\alpha) : \alpha < \omega_1\}$ of $\mathcal{F} \times \omega$. Given $F \in \mathcal{F}$, choose $\sigma_F = \max\{\sigma_x : x \in F\}$. We shall recursively build sequences $\{f_\alpha : \alpha < \omega_1\} \subset A$ and $\{x_\alpha : \alpha < \omega_1\} \subset X$ as follows.

Fix $x_0 \in X \setminus F_0$ satisfying $\sigma_{F_0} < \sigma_{x_0}$, and fix $f_0 \in A$ such that $|f_0(x)| < 1/2^{n_0}$ for each $x \in F_0$ and $f_0(x_0) > 1$. Assume that for some ordinal $0 < \alpha < \omega_1$ we have constructed $\{f_\beta : \beta < \alpha\} \subset A$ and $\{x_\beta : \beta < \alpha\} \subset X$ satisfying, for each $\gamma < \beta < \alpha$, the following properties:

(1) if $\sigma_\gamma = \max\{\sigma_{x_\gamma}, \sigma_{f_\gamma}, \sigma_{F_\gamma}\}$, then $p_{\sigma_\gamma}(x_\beta) = p_{\sigma_\gamma}(x_\gamma)$;
(2) $\sigma_{F_\beta} < \sigma_{x_\beta}$ and $\sigma_\gamma < \sigma_{x_\beta} \leq \sigma_\beta$;
(3) $|f_\beta(X)| < 1/2^{n_\beta}$ for each $x \in F_\beta$ and $f_\beta(x_\beta) > 1$.

Now, observe that $\{p_{\sigma_\beta}(x_\beta) : \beta < \alpha\}$ is a sequence of compatible functions, so we can choose $x_\alpha \in X$ such that

$$\sigma_{x_\alpha} > \max\{\sup\{\sigma_\beta : \beta < \alpha\}, \sigma_{F_\alpha}\}$$

and $p_{\sigma_\beta}(x_\alpha) = p_{\sigma_\beta}(x_\beta)$ for all $\beta < \alpha$. Note that $\sigma_{F_\alpha} < \sigma_{x_\alpha}$ implies that $x_\alpha \notin F_\alpha$. Choose $f_\alpha \in A$ such that $|f_\alpha(X)| < 1/2^{n_\alpha}$ for each $x \in F_\alpha$ and $f_\alpha(x_\alpha) > 1$. Then the sequences $\{f_\beta : \beta < \alpha + 1\} \subset A$ and $\{x_\beta : \beta < \alpha + 1\} \subset X$ also satisfy the properties (1)-(3). This finishes the construction.

Observe that the point $x = \bigcup\{p_{\sigma_\alpha}(x_\alpha) : \alpha < \omega_1\} \in K$ is well defined. We know that $\psi_x \restriction_C$ is continuous for every countable set $C \subset C_p(X)$, so to see that A does not have countable functional tightness, we will prove that $\psi_x \restriction_A$ is discontinuous. On the one hand, note that the continuous extension $\hat{\zeta}$ of ζ to K is the zero constant function on K and so $\psi_x(\zeta) = \varphi_x(\hat{\zeta}) = \hat{\zeta}(X) = 0$. On the other hand, for each $\alpha < \omega_1$, since $\sigma_{f_\alpha} \leq \sigma_\alpha$, we can choose a continuous function $g_\alpha : 2^{\sigma_\alpha} \to \mathbb{R}$ such that $f_\alpha = g_\alpha \circ p_{\sigma_\alpha} \restriction_X$. We then have $\hat{f}_\alpha = g_\alpha \circ p_{\sigma_\alpha}$ and hence $\psi_x(f_\alpha) = \varphi_x(\hat{f}_\alpha) = \hat{f}_\alpha(x) = g_\alpha(p_{\sigma_\alpha}(x)) = g_\alpha(p_{\sigma_\alpha}(x_\alpha)) = f_\alpha(x_\alpha) > 1$. Thus, to conclude that ψ_x is discontinuous, it suffices to verify that ζ belongs to the closure of $B := \{f_\alpha : \alpha < \omega_1\}$. Given an arbitrary $F \in \mathcal{F}$ and $\varepsilon > 0$, we can choose $n < \omega$ such that $1/2^n < \varepsilon$ and $0 < \alpha < \omega_1$ such that $(F_\alpha, n_\alpha) = (F, n)$. Then the choice of f_α implies that $|f_\alpha(x)| < 1/2^n < \varepsilon$ for each $x \in F$; this shows that $\zeta \in \operatorname{cl}(B)$. □

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5. The Lindelöf property in function spaces over Alexandroff doubles

In this section, we characterize countably compact spaces for which their Alexandroff double has a Lindelöf C_p. As before, we need to introduce some notation.

The Alexandroff double $AD(X)$ of a space X is the space $X \times 2$ with the topology in which all points in $X \times \{1\}$ are isolated, and basic neighborhoods of points $\langle x, 0\rangle$ are of the form $(U \times 2) \setminus \{\langle x, 1\rangle\}$ where U is a neighborhood of x in X. We canonically identify X with $X \times \{0\}$, choose $X' = X \times \{1\}$ and denote by r the canonical projection from $AD(X)$ onto X.

Given a set $A \subset X$ and a point $x \in X$, we take $C_{p,A}(X) = \{f \in C_p(X) : f \restriction_A \equiv 0\}$ and $C_{p,x}(X) = C_{p,\{x\}}(X)$.

Proposition 5.1. *If $X \times \omega^\omega$ is Lindelöf and Y is K-analytic, then $X \times Y$ is Lindelöf.*

Proof. Since Y is K-analytic, there exists a space Z that admits a perfect onto function $p : Z \to \omega^\omega$ and a continuous onto function $f : Z \to Y$. Let $i : X \to X$ be the identity function and observe that $i \times p : X \times Z \to X \times \omega^\omega$ is perfect and onto, and note that the function $i \times f : X \times Z \to X \times Y$ is continuous and onto. The Lindelöf property is preserved under perfect preimages and continuous images, so we conclude that $X \times Y$ is Lindelöf. □

Theorem 5.2. *If X is a countably compact space, then $C_p(AD(X))$ is Lindelöf if and only if $C_p(X) \times \omega^\omega$ is Lindelöf.*

Proof. Consider the dual function $r^* : C_p(X) \to C_p(AD(X))$ of the retraction r and observe that r^* is a continuous linear extender. Since X is closed in $AD(X)$, we can apply Proposition 6.6.6 in [8] to see that $C_p(AD(X))$ is homeomorphic to the product $C_p(X) \times C_{p,X}(AD(X))$. Given a function $f \in C_{p,X}(AD(X))$ and $n < \omega$ choose $\mathrm{supp}_n(f) := \{x' \in X' : |f(x')| \in [1/2^n, \infty)\}$. We assert that the set $\mathrm{supp}_n(f)$ must be finite. If the set $\mathrm{supp}_n(f)$ is infinite, then the countable compactness of X implies that the infinite set $r(\mathrm{supp}_n(f)) \subset X$ has an accumulation point $z \in X$. The definition of the topology of $AD(X)$ implies that z is also an accumulation point of $\mathrm{supp}_n(f)$. The continuity of $|f|$ implies that $|f(z)| \in \mathrm{cl}\{|f(x')| : x' \in \mathrm{supp}_n(f)\} \subset [1/2^n, \infty)$, which contradicts the election of $f \in C_{p,X}(AD(X))$. Let

$$Z = \{f \in C_p(X') : \forall n < \omega\ (|\mathrm{supp}_n(f)| < \omega)\}.$$

On the one hand, we can see that $\pi_{X'} \restriction_{C_{p,X}(AD(X))}: C_{p,X}(AD(X)) \to Z$ is a homeomorphism. Indeed, its inverse is the continuous linear extender $\varphi : Z \to C_{p,X}(AD(X))$ defined as $\varphi(g) = g \cup \zeta$, where ζ is the zero constant function on X. On the other hand, if κ is the cardinality of X, we can apply Problem 105 in [13] to see that Z is homeomorphic to the space $C_p(A(\kappa))$, where $A(\kappa)$ is the one-point compactification of the discrete space of cardinality κ. Therefore, we conclude that $C_p(AD(X))$ is homeomorphic to $C_p(X) \times C_p(A(\kappa))$.

Now, we proceed with the proof of the theorem. Assume first that $C_p(X) \times \omega^\omega$ is Lindelöf. It is well known that $A(\kappa)$ is an Eberlein compact space. It follows from Problem 022 in [13] that $C_p(A(\kappa))$ is K-analytic. Then, we can apply Proposition 5.1 to conclude that $C_p(X) \times C_p(A(\kappa))$ is Lindelöf. The space $C_p(AD(X))$ is homeomorphic to $C_p(X) \times C_p(A(\kappa))$ and consequently is Lindelöf. For the converse, assume that $C_p(AD(X))$ is Lindelöf. Then $C_p(X) \times C_p(A(\kappa))$ is Lindelöf. Problem 022 in [14] and Problem 105 in [13] imply that $C_p(A(\kappa))$ is homeomorphic to $C_p(A(\kappa))^\omega$. It follows that $C_p(X) \times C_p(A(\kappa))^\omega$ is Lindelöf. The space $C_p(A(\kappa))$ is not pseudocompact and consequently it admits a closed copy of ω. It follows that ω^ω can be embedded as a closed subspace in $C_p(A(\kappa))^\omega$. So, the space $C_p(X) \times \omega^\omega$ can be embedded as a closed subspace in $C_p(X) \times C_p(A(\kappa))^\omega$ and consequently is Lindelöf. □

Corollary 5.3. *It X is compact and zero dimensional, then $C_p(AD(X))$ is Lindelöf if and only if $C_p(X)$ is Lindelöf.*

Proof. Assume that $C_p(X)$ is Lindelöf. Then we can apply Theorem 2.2 1.1 in [10] to see that $C_p(X)^\omega$ is Lindelöf. Hence $C_p(\bigoplus\{X : n < \omega\})$ is Lindelöf, where $\bigoplus\{X : n < \omega\}$ is the free sum of countable many copies of the space X. Since $\bigoplus\{X : n < \omega\}$ is not pseudocompact, we can apply Proposition 1.1 of [10] to see that $C_p(\bigoplus\{X : n < \omega\}) \times \omega^\omega$ is homeomorphic to a closed subspace of $C_p(\bigoplus\{X : n < \omega\})$ and consequently is Lindelöf. It follows that $C_p(X)^\omega \times \omega^\omega$ is Lindelöf and in particular $C_p(X) \times \omega^\omega$ is Lindelöf. An application of Theorem 5.2 implies that $C_p(AD(X))$ is Lindelöf. The converse follows directly from Theorem 5.2. □

Now we prove that under $MA + \neg CH$ the characterization obtained in Corollary 5.3 holds for arbitrary compact spaces.

Proposition 5.4. *If X contains a copy of $A(\omega)$, then $C_p(X)$ contains a closed copy of $C_p(X) \times \omega^\omega$.*

Proof. Let K be a compact subspace of X homeomorphic to $A(\omega)$. Since K is compact and metrizable, we can apply Proposition 6.6.5 from [8] to see that there exists a continuous linear extender $\varphi : C_p(K) \to C_p(X)$. The space K is closed in X, so we can apply Proposition 6.6.6 in [8] to see that $C_p(X)$ is homeomorphic to the product $C_{p,K}(X) \times C_p(K)$ and consequently is homeomorphic to the product $C_{p,K}(X) \times C_p(A(\omega))$. Problem 022 in [14] and Problem 105 in [13] imply that $C_p(A(\omega))$ is homeomorphic to $C_p(A(\omega))^\omega$. Then $C_p(X)$ is homeomorphic to $C_{p,K}(X) \times C_p(A(\omega))^\omega$ and thus it is homeomorphic to $C_p(X) \times C_p(A(\omega))^\omega$. The space $C_p(A(\omega))$ is not pseudocompact and so it admits a closed copy of ω. It follows that $C_p(A(\kappa))^\omega$ has a closed copy of ω^ω. Hence, the space $C_p(X) \times \omega^\omega$ can be embedded as a closed subspace in $C_p(X) \times C_p(A(\omega))^\omega$. Therefore, we conclude that the space $C_p(X) \times \omega^\omega$ can be embedded as a closed subspace in $C_p(X)$. □

Theorem 5.5. *Under $MA + \neg CH$, if X is compact, then $C_p(AD(X))$ is Lindelöf if and only if $C_p(X)$ is Lindelöf.*

Proof. If $C_p(X)$ is Lindelöf, then we can apply a Reznichenko's Theorem, see Problem 80 in [13], to see that X is ω-monolithic. The space X is compact, so the closure of every countable subset of X is metrizable. We can assume that X is infinite, because otherwise the result is immediate. Let A be a countable infinite subset of X. Since X is compact, A has an accumulation point $x \in X$. The closure of A is metrizable, so we can find a sequence $\{x_n : n < \omega\} \subset A$ that converges to x. Then $K = \{x\} \cup \{x_n : n < \omega\} \subset X$ is a compact subspace of X homeomorphic to $A(\omega)$. Then we can apply Proposition 5.4 to see that $C_p(X)$ contains a closed copy of $C_p(X) \times \omega^\omega$. It follows that $C_p(X) \times \omega^\omega$ is Lindelöf. Finally, we can apply Theorem 5.2 to conclude that $C_p(AD(X))$ is Lindelöf. The converse follows from Theorem 5.2. □

Now, we analyze the Lindelöf property in function spaces over some one-point extensions of free topological sums. Let us introduce the following notation.

Let X_n be a topological space for all $n < \omega$ and assume that ∞ is not an element of the free topological sum $\bigoplus\{X_n : n < \omega\}$. We consider the topology on the set $X = \bigoplus\{X_n : n < \omega\} \cup \{\infty\}$ defined as follows. Each X_i is clopen in X and keeps its original topology. The neighborhood base at ∞ consists of all sets of the form $\bigcup\{X_n : m \leq n < \omega\} \cup \{\infty\}$, where $m < \omega$.

Consider the space $X = \bigoplus\{X_n : n < \omega\} \cup \{\infty\}$. In [6], it was proved that if $C_p(\bigoplus\{X_m : m < n\}, \mathbb{Z})$ is Lindelöf for all n, then $C_p(X, \mathbb{Z})$ is Lindelöf. For the space $C_p(X)$ we obtain the following result.

Theorem 5.6. *If $X = \bigoplus\{X_n : n < \omega\} \cup \{\infty\}$, where each X_n is a compact space, then $C_p(X)$ is Lindelöf if and only if $C_p(\bigoplus\{X_n : n < \omega\})$ is Lindelöf.*

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Proof. Fix $x_n \in X_n$ for each $n < \omega$ and let $K = \{x_n : n < \omega\} \cup \{\infty\}$. Given $n < \omega$ let $J_n = [-1/2^n, 1/2^n]$, and let $J = J_0$. Consider the spaces $Y = \bigoplus\{X_n : n < \omega\}$, $Z = \{f \in C_p(X) : \forall n < \omega\ (f \restriction_{X_n} \in C_p(X_n, J_n))\}$ and $W = \prod\{C_p(X_n, J) : n < \omega\}$. Observe that Z is a closed subspace of $C_p(X)$, $\pi_Y(Z)$ is a closed subspace of $C_p(Y)$ and $\pi_Y \restriction_Z : Z \to \pi_Y(Z)$ is a homeomorphism. In addition, note that the spaces Z and W are homeomorphic.

Assume that $C_p(X)$ is Lindelöf. Note that K is homeomorphic to $A(\omega)$. Then we can apply Proposition 5.4 to see that $C_p(X)$ contains a closed copy of $C_p(X) \times \omega^\omega$. It follows that $C_p(X) \times \omega^\omega$ is Lindelöf. Since Z is closed in $C_p(X)$ and homeomorphic to W, the spaces $Z \times \omega^\omega$ and $W \times \omega^\omega$ are also Lindelöf. Given $n < \omega$, since X_n is compact, the function $\varphi_n : C_p(X_n, J) \times \omega \to C_p(X_n)$ defined as $\varphi_n(\langle f, t\rangle) = tf$ is continuous and onto. So, the function $\varphi = \prod\{\varphi_n : n < \omega\}$ is a continuous function from $W \times \omega^\omega$ onto $\prod\{C_p(X_n) : n < \omega\}$. It follows that the space $\prod\{C_p(X_n) : n < \omega\}$ is Lindelöf. The space $C_p(Y)$, which is homeomorphic to $\prod\{C_p(X_n) : n < \omega\}$, is also Lindelöf.

To prove the converse, assume that $C_p(Y)$ is Lindelöf. Since Y is not pseudocompact, we can apply Proposition 1.1 of [10] to see that $C_p(Y) \times \omega^\omega$ is homeomorphic to a closed subspace of $C_p(Y)$ and hence is Lindelöf. Since W is homeomorphic to the closed subspace $\pi_Y(Z)$ of $C_p(Y)$, the space $W \times \omega^\omega$ also is Lindelöf. The space K is homeomorphic to $A(\omega)$, so Problem 022 in [13] implies that $C_p(K)$ is K-analytic. Then, we can apply Proposition 5.1 to conclude that $W \times C_p(K)$ is Lindelöf. The space $C_{p,\infty}(K)$ is closed in $C_p(K)$, so $W \times C_{p,\infty}(K)$ is also Lindelöf. We can apply Corollary 6.6.7 in [8] to see that $C_p(X)$ is homeomorphic to $C_{p,\infty}(X) \times \mathbb{R}$. Since $\mathbb{R}$ is σ-compact, in order to prove that $C_p(X)$ is Lindelöf, it is sufficient to show that $C_{p,\infty}(X)$ is Lindelöf. So, to finish the proof, it suffices to verify that $C_{p,\infty}(X)$ is a continuous image of $W \times C_{p,\infty}(K)$.

Consider the function $\varphi : W \times C_{p,\infty}(K) \to C_{p,\infty}(X)$ given by the rule $\varphi(\langle F, g\rangle) = \bigoplus\{g(x_n) \cdot F(n) : n < \omega\} \cup \{(\infty, 0)\}$. Observe that φ is well defined and continuous. In fact, the space $C_{p,\infty}(X)$ is homeomorphic to its image $\pi_Y(C_{p,\infty}(X)) \subset C_p(Y)$ under the restriction function, so the space $C_{p,\infty}(X)$ can be canonically identified as a subspace of $\prod\{C_p(X_n) : n < \omega\}$. Note that under such identification $\varphi = \Delta\{\rho_n \circ (p_n \Delta \varphi_n) : n < \omega\}$ is a diagonal of continuous functions and hence continuous, where $p_n : W \to C_p(X_n, J)$ is the projection onto the n-th coordinate, $\varphi_n : C_{p,\infty}(K) \to \mathbb{R}$ is the evaluation function induced by x_n and $\rho_n : C_p(X_n, J) \times \mathbb{R} \to C_p(X_n)$ is the multiplication function. To finish the proof, we only need to verify that φ is onto. Choose a function $f \in C_{p,\infty}(X)$. Then we can find a strictly increasing sequence of positive natural numbers $\{N_n : n < \omega\}$ such that $f \restriction_{X_m} \in (1/2^m)\, C_p(X_m, J)$ for each $m \geq N_n$. Choose $M_n = \|f \restriction_{X_n}\| + 1$ for each $n < \omega$. Define $F \in W$ and $g \in C_{p,\infty}(K)$ as follows:

$$F(n) = \begin{cases} 2^n \cdot f \restriction_{X_n} & \text{if } n \in [N_n, N_{n+1}); \\ (1/M_n) \cdot f \restriction_{X_n} & \text{otherwise.} \end{cases}$$

$$g(x_n) = \begin{cases} 1/2^n & \text{if } n \in [N_n, N_{n+1}); \\ M_n & \text{otherwise.} \end{cases}$$

Observe that $\langle F, g\rangle \in W \times C_{p,\infty}(K)$ and $\varphi(\langle F, g\rangle) = f$; this proves that φ is onto. □

6. The Lindelöf property in function spaces over Hattori spaces

Now, we will prove that, consistently, functions spaces over Hattori spaces have the Lindelöf property only in trivial cases. In [7] Hattori defines a family of intermediate topologies between the Euclidean and Sorgenfrey topologies on the real line using local bases as follows. Fix a subset $A \subset \mathbb{R}$. Given $x \in \mathbb{R}$ and $\epsilon > 0$ consider the set

$$U_\epsilon(x) = \begin{cases} (x - \epsilon, x + \epsilon) & \text{if } x \in A; \\ ([x, x + \epsilon) & \text{if } x \in \mathbb{R} \setminus A. \end{cases}$$

The Hattori space $H(A)$ is the real line endowed with the topology in which, for each $x \in \mathbb{R}$, a local base at x consists of all sets of the form $U_\epsilon(x)$, where $\epsilon > 0$.

Recently, it was proved in [19] that in the Solovay model the space $C_p(H(A))$ is Lindelöf if and only if $\mathbb{R} \setminus A$ is countable. However, the Axiom of Choice does not hold in the Solovay model. We prove that the statement $C_p(H(A))$ is Lindelöf if and only if $\mathbb{R} \setminus A$ is countable is consistent with the Axiom of Choice, in fact such a statement holds under the Open Coloring Axiom.

Proposition 6.1. *If $a, b \in \mathbb{R} \setminus A$ and the set $[a, b) \setminus A$ is uncountable, then there exists an uncountable subspace C of the Sorgenfrey line which can be embedded in both $C_p([a, b))$ and $C_p(H(A))$ as a closed subspace.*

Proof. For simplicity, we first assume that $a = 0$ and $b = 1$. Consider the set $S = [0, 1)$ endowed with the topology inherited as a subspace of the Sorgenfrey line $\mathbb{S} = H(\emptyset)$. For each $t \in (0, 1]$ let $f_t = \chi_{[0,t)} : S \to \mathbb{R}$ be the characteristic function of $[0, t)$. Consider the function $\varphi : S \to C_p(S)$ given by $\varphi(t) = f_{1-t}$ for each $t \in S$. It was proved in Facts 1 and 2 in the proof of Problem 165 (v) in [11] that the function φ embeds S as a closed subspace in $C_p(S)$.

Denote by J the set $[0, 1)$ endowed with the topology inherited form $H(A)$. Then we have $C_p(J) \subset C_p(S)$. Since $[0, 1) \setminus A$ is uncountable, the sets $B = [(0, 1) \setminus A] \cup \{1\} \subset (0, 1]$ and $C = 1 - B = \{1 - t : t \in B\} \subset J$ are also uncountable. Given $t \in (0, 1]$, note that $f_t \in C_p(J)$ if and only if $t \in B$. It follows that, for each $t \in S$, we have that $\varphi(t) \in C_p(J)$ if and only if $1 - t \in B$ if and only if $t \in C$. Hence, we conclude that $\varphi(S) \cap C_p(J) = \varphi(C)$. This last equality implies that $\varphi \restriction_C : C \to C_p(J)$ embeds C in $C_p(J)$ as a closed subspace, where C is considered with the subspace topology of the Sorgenfrey line.

On the other hand, since $0, 1 \in \mathbb{R} \setminus A$, the set J is closed and open in $H(A)$. It follows that $H(A)$ is homeomorphic to $J \oplus (H(A) \setminus J)$ and so $C_p(H(A))$ is homeomorphic to $C_p(J) \times C_p(H(A) \setminus J)$. Since C can be embedded in $C_p(J)$ as a closed subspace, C can also be embedded in $C_p(H(A))$ as a closed subspace. This finishes the proof for this case.

When a and b are arbitrary, we can choose an order isomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(a) = 0$ and $f(b) = 1$. It follows that $f : H(A) \to H(f(A))$ is an homeomorphism. As a consequence, the dual function $f^* : C_p(H(f(A))) \to C_p(H(A))$ and its restriction $f^* : C_p([0, 1)) \to C_p([a, b))$ are homeomorphisms, so the result follows from the above case. □

Theorem 6.2. *The space $H(A)$ is Lindelöf Σ if and only if $\mathbb{R} \setminus A$ is countable.*

Proof. If $\mathbb{R} \setminus A$ is countable, then the space $H(A) = A \cup (\mathbb{R} \setminus A)$ is the union of two Lindelöf Σ-spaces and hence it is a Lindelöf Σ-space. To prove the converse, assume that $H(A)$ is Lindelöf Σ. Then we can apply Problems 268 and 154 in [12] to see that the space $C_p(H(A))$ is ω-monolithic. If $\mathbb{R} \setminus A$ is uncountable, Lemma 6.1 implies that an uncountable subspace C of the Sorgenfrey line embeds in $C_p(H(A))$ as a closed subspace. Since ω-monolithity is inherited by arbitrary subspaces, C must be ω-monolithic. The space C is separable and, being ω-monolithic, must be cosmic. On the other hand, since C is not countable, it is well known that C cannot be cosmic. This contradiction shows that $\mathbb{R} \setminus A$ is countable. □

Corollary 6.3. *A subspace A of the Sorgenfrey line is Lindelöf Σ if and only if it is countable.*

Proof. If A is a Lindelöf Σ-subspace of the Sorgenfrey line, then the space $H(\mathbb{R} \setminus A) = (\mathbb{R} \setminus A) \cup A$ is the union of two Lindelöf Σ-spaces and hence a Lindelöf Σ-space. Therefore, Theorem 6.2 that $\mathbb{R} \setminus A$ is countable. Since any countable set is Lindelöf Σ, the converse is always true. □

Corollary 6.4. *The space $C_p(H(A))$ is Lindelöf Σ if and only if $\mathbb{R} \setminus A$ is countable.*

Proof. If $\mathbb{R} \setminus A$ is countable, then $H(A) = A \cup (\mathbb{R} \setminus A)$ is the union of two cosmic spaces and hence cosmic. It follows that $C_p(H(A))$ is cosmic and, in particular, $C_p(H(A))$ is a Lindelöf Σ-space. On the other hand, if $C_p(X)$ is Lindelöf Σ, then we can apply Problem 206 in [13] to see that the realcompactification $\upsilon H(A)$ of $H(A)$ is Lindelöf Σ. Since $H(A)$ is Lindelöf, it is realcompact, and so $H(A) = \upsilon H(A)$ is Lindelöf Σ. It follows from Theorem 6.2 that $\mathbb{R} \setminus A$ is countable. □

Theorem 6.5. *Under OCA, the space $C_p(H(A))$ is Lindelöf if and only if $\mathbb{R} \setminus A$ is countable.*

Proof. If $\mathbb{R} \setminus A$ is countable, then Corollary 6.4 implies that $C_p(H(A))$ is Lindelöf.

To prove the converse, assuming that $\mathbb{R} \setminus A$ is uncountable, we will prove that $C_p(H(A))$ is not Lindelöf. Observe that we can choose $a, b, a', b' \in \mathbb{R} \setminus A$ satisfying that, if $J = [a, b)$ and $J' = [a', b')$, then $J \cap J' = \emptyset$ and the sets $J \setminus A$ and $J' \setminus A$ are uncountable. By Lemma 6.1, we can find uncountable subspaces C and C' of the Sorgenfrey line which can be embedded in $C_p(J)$ and $C_p(J')$ as closed subspaces, respectively. Let $B = H(A) \setminus (J \cup J')$. Observe that J and J' are closed and open subsets of $H(A)$, so $H(A)$ is homeomorphic to $B \oplus J \oplus J'$. It follows that $C_p(H(A))$ is homeomorphic to $C_p(B) \times C_p(J') \times C_p(J')$. Since C can be embedded in $C_p(J)$ as a closed subspace and C' can be embedded in $C_p(J')$ as a closed subspace, we deduce that $C \times C'$ can be embedded in both $C_p(J') \times C_p(J')$ and $C_p(H(A))$ as a closed subspace. To finish the proof, it suffices to show that the product $C \times C'$ is not Lindelöf.

It was proved in [18] that, under Open Coloring Axiom OCA, if X and Y are two uncountable subsets of the real line, then there exists a strictly increasing function from an uncountable subset of X into Y. It follows that there exists a strictly decreasing function f from an uncountable subset D of C into C'. It is not difficult to verify that f is a closed discrete subspace of $C \times C'$. Therefore, $C \times C'$ is not Lindelöf. □

Regarding the Lindelöf property and Hattori spaces, it is natural to establish when their finite or countable powers have the Lindelöf property. In first place, we can observe that Hattori spaces, being continuous images of the Sorgenfrey line, are always hereditarily Lindelöf. In the second place, our last result presents a characterization of the Lindelöf property in squares of Hattori spaces.

Proposition 6.6. *If X is hereditarily Lindelöf and Y is cosmic, then $X \times Y$ is hereditarily Lindelöf.*

Proof. Fix a countable network $\mathcal{N}$ for the space Y consisting of closed subsets. Choose an arbitrary subspace Z of $X \times Y$ and let $\mathcal{U}$ be an arbitrary open cover of Z. For each $z \in Z$ fix $\varphi(z) \in \mathcal{U}$ satisfying that $z \in \varphi(z)$. Denote by $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the corresponding projections. For each $z \in Z$, fix $U_z \in \mathcal{U}$ and $N_z \in \mathcal{N}$ such that $z \in U_z \times N_z \subset \varphi(z)$. Given $N \in \mathcal{N}$, let $Z_N = \{z \in Z : N_z = N\}$. Since $p_X(Z_N)$ is Lindelöf and $\{U_z : z \in Z_N\}$ is an open cover of $p_X(Z_N)$, we can find a countable subset $B_N \subset Z_N$ such that $p_X(Z_N) \subset \{U_z : z \in B_N\}$. Since $p_Y(z) \in N_z = N$ for each $z \in Z_N$, we conclude $Z_N \subset \bigcup\{U_z \times N : z \in B_N\} \subset \bigcup\{\varphi(z) : z \in B_N\}$. It follows that for the countable set $B = \bigcup\{B_N : N \in \mathcal{N}\}$ we have $Z = \bigcup\{Z_N : N \in \mathbb{N}\} \subset \bigcup\{\varphi(z) : z \in B\}$. This shows that $X \times Y$ is a Lindelöf space. □

Theorem 6.7. *The space $H(A)^2$ is Lindelöf if and only if $\mathbb{R} \setminus A$ does not contain any subspace of $\mathbb{R}$ homeomorphic to 2^ω.*

Proof. If $\mathbb{R} \setminus A$ contains a copy C of 2^ω, then let C_0 be the set of non-isolated points of C. Theorem 4.3 of [5] shows that every closed subspace of the Sorgenfrey line, which is dense in itself, is homeomorphic to the Sorgenfrey line. So C_0 is homeomorphic to $\mathbb{S}$ and, in particular, C_0^2 is not Lindelöf. We know that C_0 is closed in C and C is closed in $H(A)$, so C_0 is closed in $H(A)$ and C_0^2 is closed in $H(A)^2$. Since C_0^2 is not Lindelöf, the space $H(A)^2$ cannot be Lindelöf.

To prove the converse, assume that $\mathbb{R} \setminus A$ does not contain a subspace of $\mathbb{R}$ homeomorphic to 2^ω. Proceeding by contradiction, assume that $H(A)^2$ is not Lindelöf and let $\mathcal{U}$ be an open cover of $H(A)^2$

that does not admit countable subcovers. Proposition 6.6 implies that the space $H(A)^2 \setminus (\mathbb{R} \setminus A)^2 = (H(A) \times A) \cup (A \times H(A))$ is Lindelöf, so we can choose a countable family $\mathcal{U}_0$ of $\mathcal{U}$ that covers $H(A)^2 \setminus (\mathbb{R} \setminus A)^2$. Since $H(A)^2$ is not Lindelöf, we deduce that $H(A)^2 \setminus \bigcup \mathcal{U}_0$ cannot be Lindelöf and, in particular, it is uncountable. Observe that $H(A)^2 \setminus \bigcup \mathcal{U}_0$ is an uncountable Borel set in $\mathbb{R}^2$ and consequently contains a copy C of the cantor set. Note that $C \subset (\mathbb{R} \setminus A)^2$ and so the projection of K in some coordinate is an uncountable compact subset of $\mathbb{R}$ contained in $\mathbb{R} \setminus A$. It follows that $\mathbb{R} \setminus A$ contains a subspace of $\mathbb{R}$ homeomorphic to 2^ω, which by hypothesis is not possible. Therefore, $H(A)^2$ is Lindelöf. □

7. Open problems

The following list presents the goals not achieved while we were working on this paper.

Problem 7.1. Is it true that if X is a compact space of cardinality ω_1, then $C_p(C_p(X))$ admits a dense functionally countable subspace?

Problem 7.2. Is there a ZFC example of a space X such that $C_p(X)$ does not admit dense subspaces of countable functional tightness?

Problem 7.3. Is there a pseudocompact space X such that $C_p(X)$ is Lindelöf, but such that $C_p(AD(X))$ is not Lindelöf?

Problem 7.4. Is it true that if $C_p(X)$ is Lindelöf, then $C_p(X) \times \omega^\omega$ is Lindelöf? What happens if X is compact?

Problem 7.5. Is it true that if X is compact and $C_p(X)$ is Lindelöf, then X contains a copy of $A(\omega)$?

Problem 7.6. If in the construction of the space $\bigoplus\{X_n : n < \omega\} \cup \{\infty\}$ we replace the Frechét filter with an arbitrary filter over ω, determine under which conditions the space $C_p(\bigoplus\{X_n : n < \omega\} \cup \{\infty\})$ has the Lindelöf property.

Problem 7.7. Is it consistent with ZFC that $C_p(H(A))$ is normal if and only if $\mathbb{R} \setminus A$ is countable?

Problem 7.8. Characterize Hattori spaces $H(A)$ for which $H(A)^3$, $H(A)^n$ or $H(A)^\omega$ are Lindelöf.

Problem 7.9. Is it true that $H(A)^n$ is hereditarily a D-space for every $n < \omega$?

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