Minimal groups with isomorphic tables of marks

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Isomorphism of tables of marks

Definition

Let G, Q be finite groups. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$. Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#((Q/K')^{H'}) = \#((G/K)^H)$ for all subgroups H, K of G.

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Table of marks

The square matrix whose H, K-entry is $\#((G/K)^H)$ is called the **table of marks** of G (where H, K run through all the elements in $\mathfrak{C}(G)$). Some authors define the table of marks of G as the transpose of the previous matrix (for instance, that is how GAP defines it). This matrix is defined up to an ordering of the elements of $\mathfrak{C}(G)$, so that the groups G and Q have isomorphic tables of marks if and only if it is possible to rearrange the elements of $\mathfrak{C}(G)$ and/or $\mathfrak{C}(Q)$ so that G and Q have identical tables of marks.

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Burnside ring

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G.

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Preserved attributes

An isomorphism between the tables of marks of two groups preserves many properties of the parent group and its subgroups. Here we list a few of these properties.

Theorem

Let G, Q be finite groups with isomorphic tables of marks. Let K, H denote subgroups of G, and let K', H' denote representatives in their respective conjugacy classes of subgroups under the isomorphism between their tables of marks. Then we have that:

- $G' = Q, (1_G)' = 1_Q, |G| = |G'|, |H| = |H'|, \alpha(H, K) = \alpha(H', K'), \beta(H, K) = \beta(H', K'), |N_G(H)| = |N_Q(H')|$.

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Preserved attributes II

- The subgroup H is normal in G if and only if H' is normal in Q. In this case, G/H and Q/H' have isomorphic tables of marks.
- If $K \leq H$ and at least one of these two subgroups is normal in G, then $K' \leq H'$ for any choice of K' and H'.
- If K and H are normal subgroups of G, then $(K \cap H)' = K' \cap H'$ and $(KH)' = K'H'$.
- If $G = K \times H$, then $Q = K' \times H'$, K and K' have isomorphic tables of marks, and H and H' have isomorphic tables of marks.
- If G is a p-group, then $socle(Z(G))' = socle(Z(Q))$.

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Preserved attributes III

- The subgroup H is maximal in G if and only if H' is maximal in Q.
- The Frattini subgroups correspond, that is, $\Phi(G)' = \Phi(Q)$.
- The group G is nilpotent if and only if Q is nilpotent. However, there are non-isomorphic p-groups with isomorphic tables of marks.
- For any divisor d of the order of H, the number of subgroups of H of order d is preserved; in particular, the total number of subgroups of H is preserved.
- The subgroup H is cyclic if and only if H' is cyclic.

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Preserved attributes IV

- If H is isomorphic to the quaternion group of order 8, then H' is isomorphic to H.
- If G is abelian then $G \cong Q$.
- The commutator subgroups correspond, that is, $[G, G]' = [Q, Q]$. Moreover, the abelianized groups are isomorphic, that is, $G/[G, G] \cong Q/[Q, Q]$.
- If G is isomorphic to S_n for some $n \geq 5$, then Q is isomorphic to G.
- The subgroup H is an elementary abelian p-group if and only if H' is an elementary abelian p-group.

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Invariants that are not preserved

Theorem

Let G and Q be finite groups with isomorphic tables of marks, and let $H \mapsto H'$ denote an isomorphism between their tables of marks. We have that

1. *H and H' may not be isomorphic.*
2. *Even if H is abelian, H' need not be abelian.*
3. *H and H' may have different tables of marks.*
4. *Even if $K \times L = H$, it may not be possible to find K', L' and H' such that $K' \times L' = H'$.*
5. *Even if K is normal in H, it may not be possible to choose K' and H' such that K' is normal in H'.*
6. *Given H, the table of marks does not determine which subgroup of G is the normalizer of H in G.*

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Example: two semidirect products

Let W denote the group $S_3 \times C_8$; let α be the automorphism of W given by $\alpha(\lambda, x^i) = (\lambda, x^i\delta(\lambda))$, where δ is the only nontrivial morphism from S_3 to C_8. This defines a semidirect product G of W by C_2.

Now let β be the automorphism of W given by $\beta(\lambda, x^i) = (\lambda, x^{5i}\delta(\lambda))$. Similarly, we define the group Q as the semidirect product of W and C_2 by β.

The groups G and Q are nonisomorphic groups of order 96 whose tables of marks are isomorphic. These are the smallest known example of such groups (and believed by the authors to be the minimal such example).

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Proving their minimality

Let $A(n)$ denote the number of non-abelian groups of order n up to isomorphism. Using GAP we can list the values of n and $A(n)$ for n from 2 to 95 (we omit the cases with zeroes and ones):

8: 2; 12: 3; 16: 9; 18: 3; 20: 3; 24: 12; 27: 2; 28: 2; 30: 3; 32: 44; 36: 10; 40: 11; 42: 5; 44: 2; 48: 47; 50: 3; 52: 3; 54: 12; 56: 10; 60: 11; 63: 2; 64: 256; 66: 3; 68: 3; 70: 3; 72: 44; 76: 2; 78: 5; 80: 47; 81: 10; 84: 13; 88: 9; 90: 8; 92: 2;

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Let n be a prime number, or the square of a prime number, or a number of the form pq where $p > q$ are primes and q does not divide $p - 1$. Then all groups of order n are abelian.

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$n = pq$ with $q|(p-1)$

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Let n be a number of the form pq where $p > q$ are primes and q divides $p-1$. Then there is exactly one isomorphism class of non-abelian groups of order n.

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Similarly we do other cases, until we are left with

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$n = pq$ with $q|(p-1)$

Theorem

Let n be a number of the form pq where $p > q$ are primes and q divides $p-1$. Then there is exactly one isomorphism class of non-abelian groups of order n.

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The case $A(n) = 2$

The seven possible values of n are: 8, 27, 28, 44, 63, 76, 92.

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The case $A(n) = 5$

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Orders 16, 24, 36, 40, 54, 56, 60, 81, 84, 88, 90

For each of these orders, we list the isomorphism classes of non-abelian groups which are not direct products. Depending on the individual groups, we compute number of elements, number of normal subgroups and of conjugacy classes of subgroups of a given order in order to differentiate their tables of marks.

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Remaining 5 cases

$n = 32, 48, 64, 72, 80$

There are 44 isomorphism classes of non-abelian groups of order 32, 47 non-abelian groups of order 48, 256 non-abelian groups of order 64, 44 non-abelian groups of order 72 and 47 non-abelian groups of order 80.

Work in progress...

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Final words

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Thank you!