Global representation rings

Gerardo Raggi-Cárdenas and Luis Valero-Elizondo

October 2015

Loyola Universtiy Chicago

AMS Sectional Meeting

available online at computo.fismat.umich.mx/∼valero

Character table

Definition

Throughout this lecture, let G denote a finite group. The character table of G is a square matrix, whose columns are indexed by the conjugacy classes of elements g of G, and whose rows are indexed by isomorphism classes of simple $\mathbb{C}G$-modules S, and the entry at such column and row is given by $X_S(g)$, which is the trace of the matrix by which g acts on S.

Character table

Definition

Throughout this lecture, let G denote a finite group. The character table of G is a square matrix, whose columns are indexed by the conjugacy classes of elements g of G, and whose rows are indexed by isomorphism classes of simple $\mathbb{C}G$-modules S, and the entry at such column and row is given by $X_S(g)$, which is the trace of the matrix by which g acts on S.

Character table

Definition

Throughout this lecture, let G denote a finite group. The character table of G is a square matrix, whose columns are indexed by the conjugacy classes of elements g of G, and whose rows are indexed by isomorphism classes of simple $\mathbb{C}G$-modules S, and the entry at such column and row is given by $X_S(g)$, which is the trace of the matrix by which g acts on S.

Table of marks

Definition

The table of marks of G is a square matrix, whose columns and rows are indexed by the conjugacy classes of subgroups of G, and whose entry H, K (where H and K are subgroups of G) is denoted $\varphi_H(G/K)$, called the mark of H in G/K, and equals the number of fixed points of G/K under the action of H.

Table of marks

Definition

The table of marks of G is a square matrix, whose columns and rows are indexed by the conjugacy classes of subgroups of G, and whose entry H, K (where H and K are subgroups of G) is denoted $\varphi_H(G/K)$, called the mark of H in G/K, and equals the number of fixed points of G/K under the action of H.

Table of marks

Definition

The table of marks of G is a square matrix, whose columns and rows are indexed by the conjugacy classes of subgroups of G, and whose entry H, K (where H and K are subgroups of G) is denoted $\varphi_H(G/K)$, called the mark of H in G/K, and equals the number of fixed points of G/K under the action of H.

The representation ring

Definition

Take the Grothendieck group on the characters of the simple $\mathbb{C}G$-modules (up to isomorphism). Note that the sum of characters can also be given by the direct sum of representations. Define a product using the tensor product of characters. This gives a ring structure, called the representation ring of the group G, or the Green ring of G.

The representation ring

Definition

Take the Grothendieck group on the characters of the simple $\mathbb{C}G$-modules (up to isomorphism). Note that the sum of characters can also be given by the direct sum of representations. Define a product using the tensor product of characters. This gives a ring structure, called the representation ring of the group G, or the Green ring of G.

The representation ring

Definition

Take the Grothendieck group on the characters of the simple $\mathbb{C}G$-modules (up to isomorphism). Note that the sum of characters can also be given by the direct sum of representations. Define a product using the tensor product of characters. This gives a ring structure, called the representation ring of the group G, or the Green ring of G.

The representation ring

Definition

Take the Grothendieck group on the characters of the simple $\mathbb{C}G$-modules (up to isomorphism). Note that the sum of characters can also be given by the direct sum of representations. Define a product using the tensor product of characters. This gives a ring structure, called the representation ring of the group G, or the Green ring of G.

The Burnside ring

Definition

The Burnside ring of the group G is the subring of $\mathbb{Z}^n$ (where n is the number of conjugacy classes of subgroups of G) spanned by the columns of the table of marks. Note that if two rings have the same tables of marks, then their Burnside rings are isomorphic: the converse is an open problem. The Burnside ring can alternatively be defined using the Grothendieck group on the transitive G-sets, and defining a multiplication using the cartesian product of G-sets. Any ring homomorphism f from $B(G)$ to $\mathbb{Z}$ is a mark, that is, there exists a fixed subgroup H of G such that $f(G/K) = (G/K)^H = \varphi_H(G/K)$ for all K.

The Burnside ring

Definition

The Burnside ring of the group G is the subring of $\mathbb{Z}^n$ (where n is the number of conjugacy classes of subgroups of G) spanned by the columns of the table of marks. Note that if two rings have the same tables of marks, then their Burnside rings are isomorphic: the converse is an open problem. The Burnside ring can alternatively be defined using the Grothendieck group on the transitive G-sets, and defining a multiplication using the cartesian product of G-sets. Any ring homomorphism f from $B(G)$ to $\mathbb{Z}$ is a mark, that is, there exists a fixed subgroup H of G such that $f(G/K) = (G/K)^H = \varphi_H(G/K)$ for all K.

The Burnside ring

Definition

The Burnside ring of the group G is the subring of $\mathbb{Z}^n$ (where n is the number of conjugacy classes of subgroups of G) spanned by the columns of the table of marks. Note that if two rings have the same tables of marks, then their Burnside rings are isomorphic: the converse is an open problem. The Burnside ring can alternatively be defined using the Grothendieck group on the transitive G-sets, and defining a multiplication using the cartesian product of G-sets. Any ring homomorphism f from $B(G)$ to $\mathbb{Z}$ is a mark, that is, there exists a fixed subgroup H of G such that $f(G/K) = (G/K)^H = \varphi_H(G/K)$ for all K.

The Burnside ring

Definition

The Burnside ring of the group G is the subring of $\mathbb{Z}^n$ (where n is the number of conjugacy classes of subgroups of G) spanned by the columns of the table of marks. Note that if two rings have the same tables of marks, then their Burnside rings are isomorphic: the converse is an open problem. The Burnside ring can alternatively be defined using the Grothendieck group on the transitive G-sets, and defining a multiplication using the cartesian product of G-sets. Any ring homomorphism f from $B(G)$ to $\mathbb{Z}$ is a mark, that is, there exists a fixed subgroup H of G such that $f(G/K) = (G/K)^H = \varphi_H(G/K)$ for all K.

Graded module

Definition

Let X be a finite G-set. We say that a $\mathbb{C}G$-module V is X-graded if

$$V = \bigoplus_{x \in X} V_x, V_x \neq 0$$

where we require that $gV_x = V_{gx}$ for all $x \in X$.

Graded module

Definition

Let X be a finite G-set. We say that a $\mathbb{C}G$-module V is X-graded if

$$V = \bigoplus_{x \in X} V_x, V_x \neq 0$$

where we require that $gV_x = V_{gx}$ for all $x \in X$.

Graded module

Definition

Let X be a finite G-set. We say that a $\mathbb{C}G$-module V is X-graded if

$$V = \bigoplus_{x \in X} V_x, V_x \neq 0$$

where we require that $gV_x = V_{gx}$ for all $x \in X$.

Bundle category

Definition

Let $\mathcal{L}$ be the category whose objects are pairs (X, V) where X is a finite G-set, and V is an X-graded $\mathbb{C}G$-module. The morphisms in $\mathcal{L}$ from (X, V) to (Y, W) are pairs of morphisms (α, f) where $\alpha : X \longrightarrow Y$ is a morphism of G-sets, $f : V \longrightarrow W$ is a morphism of $\mathbb{C}G$-modules and $f(V_x) \subseteq W_{\alpha(x)}$ for all x in X.

Bundle category

Definition

Let $\mathcal{L}$ be the category whose objects are pairs (X, V) where X is a finite G-set, and V is an X-graded $\mathbb{C}G$-module. The morphisms in $\mathcal{L}$ from (X, V) to (Y, W) are pairs of morphisms (α, f) where $\alpha : X \longrightarrow Y$ is a morphism of G-sets, $f : V \longrightarrow W$ is a morphism of $\mathbb{C}G$-modules and $f(V_x) \subseteq W_{\alpha(x)}$ for all x in X.

Examples

- $(\emptyset, 0)$ is the zero element
- $(\cdot, \mathbb{C})$ is the unity, where $\mathbb{C}$ denotes the trivial $\mathbb{C}$-module
- $(X \sqcup Y, V \oplus W)$ is the sum of (X, V) and (Y, W)
- $(X \times Y, V \otimes W)$ is the product of (X, V) and (Y, W)

Examples

- $(\emptyset, 0)$ is the zero element
- $(\cdot, \mathbb{C})$ is the unity, where $\mathbb{C}$ denotes the trivial $\mathbb{C}$-module
- $(X \sqcup Y, V \oplus W)$ is the sum of (X, V) and (Y, W)
- $(X \times Y, V \otimes W)$ is the product of (X, V) and (Y, W)

Examples

- $(\emptyset, 0)$ is the zero element
- $(\cdot, \mathbb{C})$ is the unity, where $\mathbb{C}$ denotes the trivial $\mathbb{C}$-module
- $(X \sqcup Y, V \oplus W)$ is the sum of (X, V) and (Y, W)
- $(X \times Y, V \otimes W)$ is the product of (X, V) and (Y, W)

Examples

- $(\emptyset, 0)$ is the zero element
- $(\cdot, \mathbb{C})$ is the unity, where $\mathbb{C}$ denotes the trivial $\mathbb{C}$-module
- $(X \sqcup Y, V \oplus W)$ is the sum of (X, V) and (Y, W)
- $(X \times Y, V \otimes W)$ is the product of (X, V) and (Y, W)

$[H, W]$

Definition

Let H be a subgroup of G and W a $\mathbb{C}H$-module. We denote the isomorphism class of the object $(G/H, i_H^G(W))$ by $[H, W]$.
Note that $[H, W] = [K, U]$ iff there exists g in G such that ${}^gH = K$ and ${}^gW \cong U$ as $\mathbb{C}K$-modules.

$[H, W]$

Definition

Let H be a subgroup of G and W a $\mathbb{C}H$-module. We denote the isomorphism class of the object $(G/H, i_H^G(W))$ by $[H, W]$.
Note that $[H, W] = [K, U]$ iff there exists g in G such that ${}^gH = K$ and ${}^gW \cong U$ as $\mathbb{C}K$-modules.

$T(G)$

Definition

From the category $\mathcal{L}$ we construct its Grothendieck group and create the ring $T(G)$. This ring has a natural grading: $T_n(G) = \langle [X, V] | \dim V_x = n \rangle$. Note that $T_1(G)$ is the monomial ring. We have $T(G) = \bigoplus_{n \geq 1} T_n(G)$ and $T_n(G) T_m(G) \subset T_{nm}(G)$.

$T(G)$

Definition

From the category $\mathcal{L}$ we construct its Grothendieck group and create the ring $T(G)$. This ring has a natural grading: $T_n(G) = \langle [X, V] | \dim V_x = n \rangle$. Note that $T_1(G)$ is the monomial ring. We have $T(G) = \bigoplus_{n \geq 1} T_n(G)$ and $T_n(G)T_m(G) \subset T_{nm}(G)$.

$T(G)$

Definition

From the category $\mathcal{L}$ we construct its Grothendieck group and create the ring $T(G)$. This ring has a natural grading: $T_n(G) = \langle [X, V] | \mathrm{dim} V_x = n \rangle$. Note that $T_1(G)$ is the monomial ring. We have $T(G) = \bigoplus_{n \geq 1} T_n(G)$ and $T_n(G) T_m(G) \subset T_{nm}(G)$.

$T(G)$

Definition

From the category $\mathcal{L}$ we construct its Grothendieck group and create the ring $T(G)$. This ring has a natural grading: $T_n(G) = \langle [X,V] | \mathrm{dim} V_x = n \rangle$. Note that $T_1(G)$ is the monomial ring. We have $T(G) = \bigoplus_{n \geq 1} T_n(G)$ and $T_n(G) T_m(G) \subset T_{nm}(G)$.

$T(G)$

Definition

From the category $\mathcal{L}$ we construct its Grothendieck group and create the ring $T(G)$. This ring has a natural grading: $T_n(G) = \langle [X,V] | \mathrm{dim} V_x = n \rangle$. Note that $T_1(G)$ is the monomial ring. We have $T(G) = \bigoplus_{n \geq 1} T_n(G)$ and $T_n(G)T_m(G) \subset T_{nm}(G)$.

Mackey formula

Theorem

We have a product formula:

$$[H,W][K,U] = \sum_{x\in[H\backslash G/K]} [H\cap {}^xK, r^H_{H\cap {}^xK}(W) r^{{}^xK}_{H\cap {}^xK}({}^xU)]$$

Mackey formula

Theorem

We have a product formula:

$$[H,W][K,U] = \sum_{x\in[H\backslash G/K]} [H\cap {}^xK, r^H_{H\cap {}^xK}(W) r^{{}^xK}_{H\cap {}^xK}({}^xU)]$$

Global representation ring $Д(G)$

Definition

We define the global representation ring of the group G as the quotient

$$Д(G) = T(G)/\left\langle [H, V_1 \oplus V_2] - [H, V_1] - [H, V_2] \right\rangle .$$

We also write $[H, V]$ to denote elements of $Д(G)$. We shall refer to the previous ideal of T(G) as I.

Global representation ring Д(G)

Definition

We define the global representation ring of the group G as the quotient

$$Д(G) = T(G)/\left\langle [H, V_1 \oplus V_2] - [H, V_1] - [H, V_2] \right\rangle .$$

We also write $[H, V]$ to denote elements of Д(G). We shall refer to the previous ideal of T(G) as I.

Basis of $Д(G)$

Definition

The elements $[H, S]$ with S simple form a basis of $Д(G)$ over the integers. They still satisfy the Mackey formula, and coincide when simultaneously conjugated (as before the quotient).

Basis of $Д(G)$

Definition

The elements $[H, S]$ with S simple form a basis of $Д(G)$ over the integers. They still satisfy the Mackey formula, and coincide when simultaneously conjugated (as before the quotient).

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X, V) = \sum_{\substack{x \in X \\ H \leq G_x}} X_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K, S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} X_{{}^xS}(b)$$

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X,V) = \sum_{\substack{x \in X \\ H \leq G_x}} \chi_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K,S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} \chi_{{}^xS}(b)$$

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X, V) = \sum_{\substack{x \in X \\ H \leq G_x}} X_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K, S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} X_{^xS}(b)$$

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X, V) = \sum_{\substack{x \in X \\ H \leq G_x}} \chi_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K, S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} \chi_{{}^x S}(b)$$

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X,V) = \sum_{\substack{x \in X \\ H \leq G_x}} X_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K,S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} X_{^xS}(b)$$

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X, V) = \sum_{\substack{x \in X \\ H \leq G_x}} X_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K, S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} X_{{}^x S}(b)$$

Ring homomorphisms from $T(G)$ to $\mathbb{C}$

Definition

Let H be a subgroup of G and b in H. We define a ring homomorphism $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ on the generators (X, V) by

$$\mathcal{S}_{H,b}(X, V) = \sum_{\substack{x \in X \\ H \leq G_x}} X_{V_x}(b),$$

this formula in the basis looks like

$$\mathcal{S}_{H,b}([K, S]) = \sum_{\substack{x \in [G/K] \\ H \leq^x K}} X_{{}^x S}(b)$$

Properties of $\mathcal{S}_{H,b}$

Theorem

$\mathcal{S}_{H,b} = \mathcal{S}_{K,c}$ iff there exists g in G such that ${}^gH = K$ and ${}^gb = c$.

$I \subseteq \mathrm{Ker}(\mathcal{S}_{H,b})$ for all (H,b)

$$I = \bigcap_{(H,b)} \mathrm{Ker}(\mathcal{S}_{H,b}).$$

Properties of $\mathcal{S}_{H,b}$

Theorem

$\mathcal{S}_{H,b} = \mathcal{S}_{K,c}$ *iff there exists* g *in* G *such that* ${}^gH = K$ *and* ${}^gb = c$.

$I \subseteq \mathrm{Ker}(\mathcal{S}_{H,b})$ *for all* (H, b)

$$I = \bigcap_{(H,b)} \mathrm{Ker}(\mathcal{S}_{H,b}).$$

Properties of $\mathcal{S}_{H,b}$

Theorem

$\mathcal{S}_{H,b} = \mathcal{S}_{K,c}$ *iff there exists* g *in* G *such that* ${}^gH = K$ *and* ${}^gb = c$.

$I \subseteq \mathrm{Ker}(\mathcal{S}_{H,b})$ *for all* (H,b)

$$I = \bigcap_{(H,b)} \mathrm{Ker}(\mathcal{S}_{H,b}).$$

Ring homomorphisms from $Д(G)$ to $\mathbb{C}$

Definition

Each $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ induces a ring homomorphism $\mathcal{S}_{H,b} : Д(G) \longrightarrow \mathbb{C}$. In fact, every ring homomorphism from $Д(G)$ to $\mathbb{C}$ is one of these.

Ring homomorphisms from $Д(G)$ to $\mathbb{C}$

Definition

Each $\mathcal{S}_{H,b} : T(G) \longrightarrow \mathbb{C}$ induces a ring homomorphism $\mathcal{S}_{H,b} : Д(G) \longrightarrow \mathbb{C}$. In fact, every ring homomorphism from $Д(G)$ to $\mathbb{C}$ is one of these.

Green biset functor

The association $G \to \text{Д}(G)$ can be extended so that it becomes a Green biset functor induced by the maps

$$\text{Д}({}_GU_H) : \text{Д}(H) \to \text{Д}(G)$$

$$\text{Д}({}_GU_H)(\overline{(X,V)}) = \overline{(U \times_H X, \mathbb{C}U \otimes_{\mathbb{C}H} V)}$$

and

$$\text{Д}(H) \times \text{Д}(G) \to \text{Д}(H \times G)$$

$$\overline{(X,V)} \times \overline{(Y,W)} \to \overline{(X \times Y, V \otimes_{\mathbb{C}} W)}$$

Green biset functor

The association $G \to \text{Д}(G)$ can be extended so that it becomes a Green biset functor induced by the maps

$$\text{Д}({}_G U_H) : \text{Д}(H) \to \text{Д}(G)$$

$$\text{Д}({}_G U_H)(\overline{(X,V)}) = \overline{(U \times_H X, \mathbb{C}U \otimes_{\mathbb{C}H} V)}$$

and

$$\text{Д}(H) \times \text{Д}(G) \to \text{Д}(H \times G)$$

$$\overline{(X,V)} \times \overline{(Y,W)} \to \overline{(X \times Y, V \otimes_{\mathbb{C}} W)}$$

Green biset functor

The association $G \to Д(G)$ can be extended so that it becomes a Green biset functor induced by the maps

$$Д({}_GU_H) : Д(H) \to Д(G)$$

$$Д({}_GU_H)(\overline{(X,V)}) = \overline{(U \times_H X, \mathbb{C}U \otimes_{\mathbb{C}H} V)}$$

and

$$Д(H) \times Д(G) \to Д(H \times G)$$

$$\overline{(X,V)} \times \overline{(Y,W)} \to \overline{(X \times Y, V \otimes_{\mathbb{C}} W)}$$

Green biset functor

The association $G \to Д(G)$ can be extended so that it becomes a Green biset functor induced by the maps

$$Д({}_G U_H) : Д(H) \to Д(G)$$

$$Д({}_G U_H)(\overline{(X,V)}) = \overline{(U \times_H X, \mathbb{C}U \otimes_{\mathbb{C}H} V)}$$

and

$$Д(H) \times Д(G) \to Д(H \times G)$$

$$\overline{(X,V)} \times \overline{(Y,W)} \to \overline{(X \times Y, V \otimes_{\mathbb{C}} W)}$$

Green biset functor

The association $G \to Д(G)$ can be extended so that it becomes a Green biset functor induced by the maps

$$Д({}_GU_H) : Д(H) \to Д(G)$$

$$Д({}_GU_H)(\overline{(X,V)}) = \overline{(U \times_H X, \mathbb{C}U \otimes_{\mathbb{C}H} V)}$$

and

$$Д(H) \times Д(G) \to Д(H \times G)$$

$$\overline{(X,V)} \times \overline{(Y,W)} \to \overline{(X \times Y, V \otimes_{\mathbb{C}} W)}$$

The plus construction

Another way to define this ring is as Boltje's plus construction.
Starting with the representation ring $R(G)$ and one gets exactly Д(G).

The plus construction

Another way to define this ring is as Boltje's plus construction. Starting with the representation ring $R(G)$ and one gets exactly $Д(G)$.

One ring automorphism on $Д(G)$

Definition

Define a ring automorphism σ on $Д(G)$ by sending $[K, V]$ to $[K, V^*]$. For every $\mathcal{S}_{H,b}$ we have that $\mathcal{S}_{H,b} \circ \sigma = \mathcal{S}_{H,b^{-1}}$.

One ring automorphism on $Д(G)$

Definition

Define a ring automorphism σ on $Д(G)$ by sending $[K, V]$ to $[K, V^*]$. For every $\mathcal{S}_{H,b}$ we have that $\mathcal{S}_{H,b} \circ \sigma = \mathcal{S}_{H,b^{-1}}$.

Marks and $\mathcal{S}_{H,b}$

Theorem

$$\mathcal{S}_{H,1}([K,S]) = \varphi_H(G/K)\mathrm{dim}S$$

$$\mathcal{S}_{H,b}([K,\mathbb{C}]) = \varphi_H(G/K)$$

Marks and $\mathcal{S}_{H,b}$

Theorem

$$\mathcal{S}_{H,1}([K,S]) = \varphi_H(G/K)\mathrm{dim}S$$

$$\mathcal{S}_{H,b}([K,\mathbb{C}]) = \varphi_H(G/K)$$

Table of species

Definition

The square table whose rows are indexed by $[H,b]$ and columns by $[K,S]$ and whose entries are $\mathcal{S}_{H,b}([K,S])$ is the table of species. This matrix consists of blocks (indexed by conjugacy classes of subgroups of G) along the diagonal, and zeros below them. The last block on the diagonal is the character table of G.

Table of species

Definition

The square table whose rows are indexed by $[H, b]$ and columns by $[K, S]$ and whose entries are $\mathcal{S}_{H,b}([K, S])$ is the table of species. This matrix consists of blocks (indexed by conjugacy classes of subgroups of G) along the diagonal, and zeros below them. The last block on the diagonal is the character table of G.

Table of species

Definition

The square table whose rows are indexed by $[H, b]$ and columns by $[K, S]$ and whose entries are $\mathcal{S}_{H,b}([K, S])$ is the table of species. This matrix consists of blocks (indexed by conjugacy classes of subgroups of G) along the diagonal, and zeros below them. The last block on the diagonal is the character table of G.

An application

Theorem

The order of G is an even integer if and only if there exists a ring homomorphism $f : Д(G) \longrightarrow \mathbb{C}$ whose image lies inside the real numbers and such that f is not a mark (that is, $f \neq S_{H,1}$ for all H).

The prime ideals of $\text{Д}(G)$

Let n be the order of the group G, and w a primitive complex n-th root of unity. For each $S_{H,b}$, let $P_{H,b}$ denote its kernel, and $Im_{H,b}$ its image inside $\mathbb{Z}[w]$. Let m denote a maximal ideal of $\mathbb{Z}[w]$, let $\overline{S^m_{H,b}}$ be the composition of $S_{H,b}$ followed by the quotient map to $\mathbb{Z}[w]/m$, and let $P_{H,b,m}$ be the kernel of $\overline{S^m_{H,b}}$. Let p denote the characteristic of $\mathbb{Z}[w]/m$.

Theorem

The spectrum of $\text{Д}(G)$ consists of all the ideals of the form $P_{H,b}$ and $P_{H,b,m}$.

The prime ideals of $\text{Д}(G)$

Let n be the order of the group G, and w a primitive complex n-th root of unity. For each $\mathcal{S}_{H,b}$, let $P_{H,b}$ denote its kernel, and $Im_{H,b}$ its image inside $\mathbb{Z}[w]$. Let m denote a maximal ideal of $\mathbb{Z}[w]$, let $\overline{\mathcal{S}^m_{H,b}}$ be the composition of $\mathcal{S}_{H,b}$ followed by the quotient map to $\mathbb{Z}[w]/m$, and let $P_{H,b,m}$ be the kernel of $\overline{\mathcal{S}^m_{H,b}}$. Let p denote the characteristic of $\mathbb{Z}[w]/m$.

Theorem

The spectrum of $\text{Д}(G)$ consists of all the ideals of the form $P_{H,b}$ and $P_{H,b,m}$.

The prime ideals of $\mathbb{J}(G)$

Let n be the order of the group G, and w a primitive complex n-th root of unity. For each $\mathcal{S}_{H,b}$, let $P_{H,b}$ denote its kernel, and $Im_{H,b}$ its image inside $\mathbb{Z}[w]$. Let m denote a maximal ideal of $\mathbb{Z}[w]$, let $\overline{\mathcal{S}^m_{H,b}}$ be the composition of $\mathcal{S}_{H,b}$ followed by the quotient map to $\mathbb{Z}[w]/m$, and let $P_{H,b,m}$ be the kernel of $\overline{\mathcal{S}^m_{H,b}}$. Let p denote the characteristic of $\mathbb{Z}[w]/m$.

Theorem

The spectrum of $\mathbb{J}(G)$ consists of all the ideals of the form $P_{H,b}$ and $P_{H,b,m}$.

The prime ideals of $\text{Д}(G)$

Let n be the order of the group G, and w a primitive complex n-th root of unity. For each $\mathcal{S}_{H,b}$, let $P_{H,b}$ denote its kernel, and $Im_{H,b}$ its image inside $\mathbb{Z}[w]$. Let m denote a maximal ideal of $\mathbb{Z}[w]$, let $\overline{\mathcal{S}^m_{H,b}}$ be the composition of $\mathcal{S}_{H,b}$ followed by the quotient map to $\mathbb{Z}[w]/m$, and let $P_{H,b,m}$ be the kernel of $\overline{\mathcal{S}^m_{H,b}}$. Let p denote the characteristic of $\mathbb{Z}[w]/m$.

Theorem

The spectrum of $\text{Д}(G)$ consists of all the ideals of the form $P_{H,b}$ and $P_{H,b,m}$.

The prime ideals of $Д(G)$

Let n be the order of the group G, and w a primitive complex n-th root of unity. For each $\mathcal{S}_{H,b}$, let $P_{H,b}$ denote its kernel, and $Im_{H,b}$ its image inside $\mathbb{Z}[w]$. Let m denote a maximal ideal of $\mathbb{Z}[w]$, let $\overline{\mathcal{S}^m_{H,b}}$ be the composition of $\mathcal{S}_{H,b}$ followed by the quotient map to $\mathbb{Z}[w]/m$, and let $P_{H,b,m}$ be the kernel of $\overline{\mathcal{S}^m_{H,b}}$. Let p denote the characteristic of $\mathbb{Z}[w]/m$.

Theorem

The spectrum of $Д(G)$ consists of all the ideals of the form $P_{H,b}$ and $P_{H,b,m}$.

The connected components

Now we can describe the connected components of the spectrum of $Д(G)$.

Theorem

Let H be a perfect subgroup of G, that is, $H = O^s(H)$. Let X_H consist of all ideals $P_{K,b}$ and $P_{K,b,m}$ where $O^s(K)$ is conjugate to H in G. Then X_H is a connected component of the spectrum of $Д(G)$, and any connected component is one of the X_H with H perfect.

The connected components

Now we can describe the connected components of the spectrum of $Д(G)$.

Theorem

Let H be a perfect subgroup of G, that is, $H = O^s(H)$. Let X_H consist of all ideals $P_{K,b}$ and $P_{K,b,m}$ where $O^s(K)$ is conjugate to H in G. Then X_H is a connected component of the spectrum of $Д(G)$, and any connected component is one of the X_H with H perfect.

The connected components

Now we can describe the connected components of the spectrum of $Д(G)$.

Theorem

Let H be a perfect subgroup of G, that is, $H = O^s(H)$. Let X_H consist of all ideals $P_{K,b}$ and $P_{K,b,m}$ where $O^s(K)$ is conjugate to H in G. Then X_H is a connected component of the spectrum of $Д(G)$, and any connected component is one of the X_H with H perfect.

The connected components

Now we can describe the connected components of the spectrum of Д(G).

Theorem

Let H be a perfect subgroup of G, that is, $H = O^s(H)$. Let X_H consist of all ideals $P_{K,b}$ and $P_{K,b,m}$ where $O^s(K)$ is conjugate to H in G. Then X_H is a connected component of the spectrum of Д(G)*, and any connected component is one of the X_H with H perfect.*

The primitive idempotents

Theorem

For every $K \leq G$ and $c \in K$, we have that

$$e_{K,c} = \frac{1}{|N_G(K) \cap C_G(c)|} \sum_{\substack{(H,S) \\ c \in H \leq K \\ S \in Irr(H)}} X_S(c)\mu(H,K)[H,S]$$

Here we take all subgroups H of G, and a set of representatives S of the irreducible $\mathbb{C}H$-modules.

We also have that the minimum positive integer t such that $te_{K,c}$ belongs to $\text{Д}_w(G) := \mathbb{Z}[w] \otimes \text{Д}(G)$ is $|N_G(K) \cap C_G(c)|$.

The primitive idempotents

Theorem

For every $K \leq G$ and $c \in K$, we have that

$$e_{K,c} = \frac{1}{|N_G(K) \cap C_G(c)|} \sum_{\substack{(H,S) \\ c \in H \leq K \\ S \in Irr(H)}} X_S(c)\mu(H,K)[H,S]$$

Here we take all subgroups H of G, and a set of representatives S of the irreducible $\mathbb{C}H$-modules.
We also have that the minimum positive integer t such that $te_{K,c}$ belongs to $D_w(G) := \mathbb{Z}[w] \otimes D(G)$ is $|N_G(K) \cap C_G(c)|$.

The primitive idempotents

Theorem

For every $K \leq G$ and $c \in K$, we have that

$$e_{K,c} = \frac{1}{|N_G(K) \cap C_G(c)|} \sum_{\substack{(H,S) \\ c \in H \leq K \\ S \in Irr(H)}} X_S(c)\mu(H,K)[H,S]$$

Here we take all subgroups H of G, and a set of representatives S of the irreducible $\mathbb{C}H$-modules.

We also have that the minimum positive integer t such that $te_{K,c}$ belongs to $\amalg_w(G) := \mathbb{Z}[w] \otimes \amalg(G)$ is $|N_G(K) \cap C_G(c)|$.

The primitive idempotents

Theorem

For every $K \leq G$ and $c \in K$, we have that

$$e_{K,c} = \frac{1}{|N_G(K) \cap C_G(c)|} \sum_{\substack{(H,S) \\ c \in H \leq K \\ S \in Irr(H)}} X_S(c)\mu(H,K)[H,S]$$

Here we take all subgroups H of G, and a set of representatives S of the irreducible $\mathbb{C}H$-modules.
We also have that the minimum positive integer t such that $te_{K,c}$ belongs to $\Pi_w(G) := \mathbb{Z}[w] \otimes \Pi(G)$ is $|N_G(K) \cap C_G(c)|$.

The primitive idempotents

Theorem

For every $K \leq G$ and $c \in K$, we have that

$$e_{K,c} = \frac{1}{|N_G(K) \cap C_G(c)|} \sum_{\substack{(H,S) \\ c \in H \leq K \\ S \in Irr(H)}} X_S(c)\mu(H,K)[H,S]$$

Here we take all subgroups H of G, and a set of representatives S of the irreducible $\mathbb{C}H$-modules.
We also have that the minimum positive integer t such that $te_{K,c}$ belongs to $D_w(G) := \mathbb{Z}[w] \otimes D(G)$ is $|N_G(K) \cap C_G(c)|$.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism. We say that that f is a species isomorphism if for every basis element $[H, S]$ in $Д(G)$ we have that $f([H, S])$ is another basis element of Д(Q), which we denote $[u(H, S), v(H, S)]$. Note that since S already depends on the subgroup H, we may just write $v(S)$ instead of $v(H, S)$ if there is no confusion.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism. We say that that f is a species isomorphism if for every basis element $[H, S]$ in $Д(G)$ we have that $f([H, S])$ is another basis element of Д(Q), which we denote $[u(H, S), v(H, S)]$. Note that since S already depends on the subgroup H, we may just write $v(S)$ instead of $v(H, S)$ if there is no confusion.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism. We say that that f is a species isomorphism if for every basis element $[H, S]$ in $Д(G)$ we have that $f([H, S])$ is another basis element of Д(Q), which we denote $[u(H, S), v(H, S)]$. Note that since S already depends on the subgroup H, we may just write $v(S)$ instead of $v(H, S)$ if there is no confusion.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism.

For every primitive idempotent $e_{H,a}$ in Д(G), we have that $f(e_{H,a})$ is a primitive idempotent of Д(Q), which we denote $e_{r(H,a),t(H,a)}$.

Theorem

For all $\mathbb{C}H$-simple modules S and all $a \in H$, we have that $r(H,a) = u(H,S)$. In particular, $u(H) := u(H,S)$ only depends on H, not on S and $r(H) := r(H,a)$ only depends on H, not on a.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism.
For every primitive idempotent $e_{H,a}$ in Д(G), we have that $f(e_{H,a})$ is a primitive idempotent of Д(Q), which we denote $e_{r(H,a),t(H,a)}$.

Theorem

For all $\mathbb{C}H$-simple modules S and all $a \in H$, we have that $r(H,a) = u(H,S)$. In particular, $u(H) := u(H,S)$ only depends on H, not on S and $r(H) := r(H,a)$ only depends on H, not on a.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism.
For every primitive idempotent $e_{H,a}$ in Д(G), we have that $f(e_{H,a})$ is a primitive idempotent of Д(Q), which we denote $e_{r(H,a),t(H,a)}$.

Theorem

For all $\mathbb{C}H$-simple modules S and all $a \in H$, we have that $r(H,a) = u(H,S)$. In particular, $u(H) := u(H,S)$ only depends on H, not on S and $r(H) := r(H,a)$ only depends on H, not on a.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism.
For every primitive idempotent $e_{H,a}$ in Д(G), we have that $f(e_{H,a})$ is a primitive idempotent of Д(Q), which we denote $e_{r(H,a),t(H,a)}$.

Theorem

For all $\mathbb{C}H$-simple modules S and all $a \in H$, we have that $r(H,a) = u(H,S)$. In particular, $u(H) := u(H,S)$ only depends on H, not on S and $r(H) := r(H,a)$ only depends on H, not on a.

Isomorphisms of species

Definition

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a ring isomorphism.
For every primitive idempotent $e_{H,a}$ in Д(G), we have that $f(e_{H,a})$ is a primitive idempotent of Д(Q), which we denote $e_{r(H,a),t(H,a)}$.

Theorem

For all $\mathbb{C}H$-simple modules S and all $a \in H$, we have that $r(H,a) = u(H,S)$. In particular, $u(H) := u(H,S)$ only depends on H, not on S and $r(H) := r(H,a)$ only depends on H, not on a.

Isomorphisms of species

Theorem

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a species isomorphism. For every subgroup H of G, f induces a group isomorphism between $C_H(N_G(H))$ and $C_{u(H)}(N_Q(u(H)))$, which is given by sending a to the element $t(a)$.

Isomorphisms of species

Theorem

Let G, Q be finite groups, and let $f : Д(G) \longrightarrow Д(Q)$ be a species isomorphism. For every subgroup H of G, f induces a group isomorphism between $C_H(N_G(H))$ and $C_{u(H)}(N_Q(u(H)))$, which is given by sending a to the element $t(a)$.

Final words

Final words

Thank you!