Minimal groups with isomorphic tables of marks (Part one)

Alberto G. Raggi-Cárdenas, Luis Valero-Elizondo and Eder Vieyra-Sánchez

September 10, 2009

Abstract

We prove that most groups of order less than 96 cannot have isomorphic tables of marks unless the groups are isomorphic. [1]

1 Introduction.

We constructed two nonisomorphic groups of order 96 with isomorphic tables of marks in [3]. According to GAP ([1]), they are the smallest groups with that property. We are now trying to prove this, that is, that if two groups of order less than 96 have isomorphic tables of marks, then they are isomorphic groups. In Section 2 we define tables of marks and list some of their properties. In Section 3 we list the possible number of nonabelian groups for each order less than 96 (we used GAP for this computation, but later we prove this rigourously). In the next Sections we prove our claim for all but 16 of these orders. In Section 8 we list the remaining cases and prove a theorem that will help us solve most of them in a later paper.

2 Tables of marks

Let G be a finite group. Let C(G) be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of C(G) are ordered non-decreasingly. The matrix whose H, K-entry is $\#(G/K)^H$ (that is, the number of fixed points of the set G/K under the action of H) is called the **table of marks** of G (where H, K run through all the elements in C(G)).

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathcal{C}(G)}$ spanned by the columns of the table of marks of G.

[1]Keywords: table of marks, group. AMS Classification: *19A22 Frobenius induction, Burnside and representation rings.*
Partially supported by CIC's Project *Mínimos grupos con tablas de marcas isomorfas* and CONACYT's Project *Funtores de tipo Burnside*

Definition 1. Let G and Q be finite groups. Let ψ be a function from C(G) to C(Q). Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#(Q/K')^{H'} = \#(G/K)^H$ for all subgroups H, K of G. We usually refer to H' as the image of H under the isomorphism of table of marks.

An isomorphism between tables of marks preserves the order of the subgroups, the order of their normalizers, it sends cyclic groups to cyclic groups and elementary abelian groups to elementary abelian groups. It also sends the derived subgroup of G to the derived subgroup of Q, maximal subgroups of G to maximal subgroups of Q, Sylow p-subgroups to Sylow p-subgroups (same p), and the Frattini subgroup of G to the Frattini subgroup of Q.

Assume now that G and Q are finite groups with isomorphic tables of marks. As we mentioned, G and Q must have the same order. It is also easy to check that if G is abelian or simple, then G and Q must be isomorphic groups. If G is a direct product, so is Q, and their corresponding factors have isomorphic tables of marks. If G is a semidirect product $N \rtimes H$ then Q is a semidirect product $N' \rtimes H$ where H and H' have isomorphic tables of marks (although we cannot say much about N and N', other than they correspond under the isomorphism of tables of marks). Proofs of these claims can be found in [2].

However, an isomorphism of tables of marks may not preserve abelian subgroups, and it may not send the centre of G to the centre of Q. This can be seen in two nonisomorphic groups of order 96 which have isomorphic tables of marks (see [3]).

3 Proving their minimality.

Let $A(n)$ denote the number of non-abelian groups of order n up to isomorphism. Using GAP we can list the values of n and $A(n)$ for n from 2 to 95:

2: 0; 3: 0; 4: 0; 5: 0; 6: 1; 7: 0; 8: 2; 9: 0; 10: 1; 11: 0; 12: 3; 13: 0; 14: 1; 15: 0; 16: 9; 17: 0; 18: 3; 19: 0; 20: 3; 21: 1; 22: 1; 23: 0; 24: 12; 25: 0; 26: 1; 27: 2; 28: 2; 29: 0; 30: 3; 31: 0; 32: 44; 33: 0; 34: 1; 35: 0; 36: 10; 37: 0; 38: 1; 39: 1; 40: 11; 41: 0; 42: 5; 43: 0; 44: 2; 45: 0; 46: 1; 47: 0; 48: 47; 49: 0; 50: 3; 51: 0; 52: 3; 53: 0; 54: 12; 55: 1; 56: 10; 57: 1; 58: 1; 59: 0; 60: 11; 61: 0; 62: 1; 63: 2; 64: 256; 65: 0; 66: 3; 67: 0; 68: 3; 69: 0; 70: 3; 71: 0; 72: 44; 73: 0; 74: 1; 75: 1; 76: 2; 77: 0; 78: 5; 79: 0; 80: 47; 81: 10; 82: 1; 83: 0; 84: 13; 85: 0; 86: 1; 87: 0; 88: 9; 89: 0; 90: 8; 91: 0; 92: 2; 93: 1; 94: 1; 95: 0;

$A(n) = 0$ for the following 40 values of n: 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 33, 35, 37, 41, 43, 45, 47, 49, 51, 53, 59, 61, 65, 67, 69, 71, 73, 77, 79, 83, 85, 87, 89, 91, 95.

$A(n) = 1$ for the following 20 values of n: 6, 10, 14, 21, 22, 26, 34, 38, 39, 46, 55, 57, 58, 62, 74, 75, 82, 86, 93, 94.

$A(n) = 2$ for the following 7 values of n: 8, 27, 28, 44, 63, 76, 92

$A(n) = 3$ for the following 9 values of n: 12, 18, 20, 30, 50, 52, 66, 68, 70.

$A(n) = 5$ for $n = 42$ and $n = 78$.

$A(n) = 8$ for $n = 90$.
$A(n) = 9$ for $n = 16$ and $n = 88$.
$A(n) = 10$ for $n = 36$, $n = 56$ and $n = 81$.
$A(n) = 11$ for $n = 40$ and $n = 60$.
$A(n) = 12$ for $n = 24$ and $n = 54$.
$A(n) = 13$ for $n = 84$.
$A(n) = 44$ for $n = 32$ and $n = 72$.
$A(n) = 47$ for $n = 48$ and $n = 80$.
$A(n) = 256$ for $n = 64$.

4 Some cases are easy:

Theorem 2. *Let n be a prime number, or the square of a prime number, or a number of the form pq where $p > q$ are primes and q does not divide $p - 1$. Then all groups of order n are abelian.*

This accounts for all the values n such that $A(n) = 0$ except for $n = 45$, which is easy to prove directly.

Theorem 3. *Let n be a number of the form pq where $p > q$ are primes and q divides $p-1$. Then there is exactly one isomorphism class of non-abelian groups of order n.*

This accounts for all the values n such that $A(n) = 1$ except for $n = 75$, which is easy to prove directly, since the only non-abelian group of order 75 must be the only non-trivial semidirect product $(C_5 \times C_5) \rtimes C_3$.

5 The case $A(n) = 2$

The seven possible values of n are: 8, 27, 28, 44, 63, 76, 92.

Here we must not only count all possible isomorphism classes of non-abelian groups, but we must also prove they have non-isomorphic tables of marks.

The case $n = 8$ is well-known (we can only have the quaternions and the dihedral group). Their tables of marks are not isomorphic, because all the subgroups of the quaternions are normal, which is not true of the dihedral group.

The case $n = 27$ is in the literature (for example, in Suzuki's introduction to the theory of groups). One of the groups has a cyclic subgroup of order 9, but the other group has no cyclic subgroups of that order.

The cases when n equals 28, 44, 76 and 92 are all of the form $4p$ with p a prime number larger than 4 and congruent to 3 modulo 4 (namely, 7, 11, 19 and 23). Here we have that the group must be a semidirect product, either $C_p \rtimes C_4$ or $C_p \rtimes (C_2 \times C_2)$. The Sylow 2-subgroups cannot correspond under an isomorphism of tables of marks.

The case $n = 63$ must be a semidirect product, either $C_7 \rtimes C_9$ or $C_7 \rtimes (C_3 \times C_3)$. The Sylow 3-subgroups cannot correspond under an isomorphism of tables of marks.

6 The case $A(n) = 3$

The nine possible values of n are: 12, 18, 20, 30, 50, 52, 66, 68, 70.

The case $n = 12$ is in the literature. The only non-abelian groups of order 12 are A_4, $S_3 \times C_2$ (which is $\cong D_{12}$) and the only non-trivial semidirect product $C_3 \rtimes C_4$. The third group has a Sylow 2-subgroup isomorphic to C_4, and the second group has a normal subgroup of order 6, so no two of these three groups have isomorphic tables of marks.

The cases when $n = 18$ and $n = 50$ are of the form $2p^2$ with $p = 3$ and $p = 5$. The group has to be either the only non-trivial semidirect product $C_{p^2} \rtimes C_2$ or one of the two non-trivial semidirect products $(C_p \times C_p) \rtimes C_2$. The first group has a cyclic Sylow p-subgroup. In the case $(C_p \times C_p) \rtimes C_2$, since $C_p \times C_p$ is a vector space over the field with p elements, its automorphisms of order 2 are easily computed: one of them has an invariant one-dimensional subspace, and the other one does not. One possible semidirect product has C_p as a direct summand, and the other one does not, so their tables of marks cannot be isomorphic.

The cases when n equals 20, 52 and 68 are all of the form $4p$ with p a prime number larger than 4 and congruent to 1 modulo 4 (namely, 5, 13 and 17). Here the possible groups are the only non-trivial semidirect product $C_p \rtimes (C_2 \times C_2)$, and the only two non-trivial semidirect products $C_p \rtimes C_4$. The Sylow 2-subgroup of the first group cannot correspond to C_4 under an isomorphism of tables of marks. In one of the semidirect products $C_p \rtimes C_4$, C_4 acts on C_p as the involution $x \mapsto x^{-1}$, so C_2 centralizes C_p, so the group has a normal subgroup of order

2; but in the other semidirect product, C_4 acts by an automorphism of C_P of order 4, so C_2 cannot centralize C_p, so this groups has no normal subgroup of order 2.

The cases when n equals 30, 66 and 70 are all of the form $2pq$ with p, q primes, $p > q > 2$ and q does not divide $p - 1$. First we observe that all groups of order less than 100 are soluble (except for A_5, which has order 60). By P. Hall's Theorem, a soluble group G of order $2pq$ has precisely one normal subgroup of order pq, so G is a semidirect product $H \rtimes C_2$ where H is a group of order pq. Since q does not divide $p-1$, H is abelian, so G is $(C_p \times C_q) \rtimes C_2$. The automorphism group of $C_p \times C_q$ has three elements of order 2, so there are at most three non-abelian choices for the group G. Precisely one of these groups has a direct factor isomorphic to C_q, and precisely another of these groups has a direct factor isomorphic to C_p, so neither two of the three groups can have isomorphic tables of marks.

7 The case $A(n) = 5$

There are two possible values for n, namely, 42 and 78. Both are numbers of the form $2pq$ with $p > q$ primes and q divides $p - 1$ (actually, $q = 3$ in both cases). Since these groups are soluble, by Hall's Theorem there is precisely one normal subgroup H of order pq. Here we have two possibilities: H could be the only non-abelian group of order pq, or H could be C_{pq} (groups from these two cases cannot have isomorphic tables of marks). If H is cyclic, there are three elements of order two in its automorphism group, so there are three possible non-trivial semidirect products $H \rtimes C_2$: one has C_p as a direct factor (but not C_q), the other has C_q as a direct factor (but not C_p), and the other has no such direct factors, so they cannot have isomorphic tables of marks.

Now assume that H is non-abelian. Note that H has a normal subgroup C_p and p subgroups C_q. An automorphism σ of H of order two must fix C_p setwise, and permute the p different C_q, so it fixes one of the C_q setwise, and acts here and on C_p either as the involution $x \mapsto x^{-1}$ or the identity. Moreover, the fixed points under σ (which form a normal subgroup of H) are trivial unless σ equals the identity map. Since H can be generated by a generator of C_p and a generator of one of the C_q's, there is only one automorphism of H of order 2, so one possible group is $H \times C_2$, and the other is the only non-trivial semidirect product $H \rtimes C_2$, and these two groups cannot have isomorphic tables of marks.

8 Remaining 16 cases

Sixteen orders remain, namely: 16, 24, 32, 36, 40, 48, 54, 56, 60, 64, 72, 80, 81, 84, 88, 90.

To solve most of these cases, in this section we prove that almost all p-subgroups with maximal cyclic subgroups can be identified in the table of marks and are therefore preserved by isomorphisms of tables of marks. The only two exceptions are $C_{p^{n-1}} \times C_p$ and $M(p^n)$ (both for p odd and $p = 2$).

The investigation of finite p-groups provides one of the most powerful methods in finite group theory. One of the best known results in the study of finite p-groups is the classification of finite non-abelian p-groups which have a cyclic maximal subgroup: for odd p there is only one such group, and for $p = 2$ there are four possible groups.

The following results are Theorems 4.1 and 4.2 from [4]. Let p denote a prime number.

Theorem 4. *Let G be a p-group of order p^n. Assume that G is nonabelian and that G has a maximal subgroup M which is cyclic. Then $n \geq 3$ and if p is odd, then G is isomorphic to the group $M(p^n)$ which has the following presentation:*

$$M(p^n) =< x, y \mid x^{pq} = y^p = 1,\ y^{-1}xy = x^{1+q} >,$$

where $q = p^{n-2}$.

If $p = 2$ and $n = 3$, then G is isomorphic to either the dihedral group or the quaternion group. If $p = 2$ and $n \geq 4$, then G is isomorphic to $M(2^n)$, the dihedral group D_g, the generalized quaternion group Q_g, or the quasi-dihedral group S_g.

Theorem 5. *The groups $G = D_{2m},\ Q_{2m},\ S_{2m}$ have the following properties:*

1. *The group Q_{2m} contains exactly one element of order 2.*
2. *The groups $D_{2m},\ Q_{2m}(m > 4)$, and S_{2m} contain two noncyclic maximal subgroups, and their isomorphism classes are (D_m, D_m), (Q_m, Q_m), and (D_m, Q_m), respectively.*

In order to tell these groups apart from $M(2^n)$ we need the following lemma, which shows that all the proper subgroups of the group $M(2^n)$ are abelian.

Lemma 6. *Let $G = M(2^n)$, and let H be a proper subgroup of G. Then H is abelian.*

Proof. Note that x^2 is in the centre of $M(2^n)$. There are two cases. First assume that $y \in H$. Since H is a proper subgroup, then $x \notin H$, so H can be generated by y and a power of x^2, which implies that H is abelian. Now assume that $y \notin H$. If H is generated by a power of x then we are finished. Assume that H contains an element of the form $x^t y$. It follows that H contains the element $x^t y x^t y$, which is a power of x. Since $x \notin H$, this element must be a power of x^2. In particular, H is contained in the subgroup generated by xy and x^2, which is abelian (since x^2 is central). □

Theorem 7. *Let G be a finite group, and let p be a prime number. Then the noncyclic p-subgroups of G which have a cyclic maximal subgroup are determined up to isomorphism by the table of marks, except that it is not possible to determine which ones are isomorphic to $C_{p^{n-1}} \times C_p$ and which ones are isomorphic to $M(p^n)$ (with $n \geq 4$).*

Proof. If p is odd, note that the only noncyclic groups of order p^n with a cyclic maximal subgroup are $C_{p^{n-1}} \times C_p$ and $M(p^n)$.

Now assume that $p = 2$. We shall use induction on the order of the 2-subgroup. Note that the only abelian 2-groups with a cyclic maximal subgroup are either cyclic or $C_{2^{n-1}} \times C_2$, and the cyclic subgroup can be determined by the table of marks. If the subgroup is nonabelian, by Theorem 4 we know that there are only four possible cases: $M(2^n)$, D_{2^n}, Q_{2^n} and S_{2^n}.

By Theorem 5, we have that Q_{2^n} is the only group which has exactly one cyclic subgroup of order 2, and this can be seen in the table of marks.

We also know that all maximal subgroups of $C_{2^{n-1}} \times C_2$ and $M(2^n)$ are abelian, whereas D_{2^n} and S_{2^n} have nonabelian maximal subgroups (which in turn have cyclic maximal subgroups): D_{2^n} has two maximal subgroups isomorphic to $D_{2^{n-1}}$, whereas S_{2^n} has two maximal subgroups isomorphic to $D_{2^{n-1}}$ and $Q_{2^{n-1}}$, which can be distinguished inductively by the table of marks (the induction starts when we can tell D_8 from Q_8 by the table of marks). □

References

[1] The GAP Group. *GAP – Groups, Algorithms and Programming, Version 4.4*, 2006. (`http:www.gap-system.org`).

[2] L. M. Huerta-Aparicio, A. Molina-Rueda, A.G. Raggi-Cárdenas, and L. Valero-Elizondo. On some invariants preserved by isomorphisms of tables of marks. To appear in Revista Colombiana de Matemáticas.

[3] Alberto G. Raggi-Cárdenas and Luis Valero-Elizondo. Two non-isomorphic groups of order 96 with isomorphic tables of marks and non-corresponding centres and abelian subgroups. *Communications in Algebra*, 37:209–212, 2009. ISSN: 0092-7872, DOI: 10.1080/00927870802243614.

[4] Michio Suzuki. *Group Theory II*, volume 248 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, 1986.