Characteristic subgroups are not preserved by isomorphisms of tables of marks

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Abstract. We construct two non-isomorphic groups G and Q of order 96 which have isomorphic tables of marks, but such that the centre of G is mapped to a non-characteristic subgroup of Q under this isomorphism of tables of marks.

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1. Introduction.

Groups with isomorphic tables of marks may not be isomorphic groups (as proved by Thévenaz in [5]), but one still expects them to have many attributes in common. Indeed, if G and Q are groups with isomorphic tables of marks, then they have isomorphic composition factors (see [2]), and they also have isomorphic Burnside rings (the converse is still an open problem, put forward also in [2]); if two groups have isomorphic Burnside rings and one of them is abelian/Hamiltonian/minimal simple, then the two groups are isomorphic (see [3]), and a similar result is known for several families of simple groups (see [1]).

It is also easy to prove that an isomorphism between tables of marks preserves normal subgroups, maximal subgroups, Sylow p-subgroups, cyclic subgroups, elementary abelian subgroups, the commutator subgroup, and the Frattini subgroup. However, it has been shown that abelian subgroups and the centres of the groups are not always preserved. In this paper we show that characteristic subgroups may not be preserved under an isomorphism between tables of marks.

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2. Tables of marks

Let G be a finite group. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. The matrix whose H, K-entry is $\#(G/K)^H$ (that is, the number of fixed points of the set G/K under the action of H) is called the **table of marks** of G (where H, K run through all the elements in $\mathfrak{C}(G)$).

The **Burnside ring** of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G.

Definition 1. Let G and Q be finite groups. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$. Given a subgroup H of G, we denote by H' any representative of $\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#(Q/K')^{H'} = \#(G/K)^H$ for all subgroups H, K of G.

3. Two non-isomorphic groups of order 96 with isomorphic tables of marks

This is a summary of [4].

Let S_3 be the symmetric group or order 6. Let C_8 be the cyclic group of order 8, generated by x, and let C_2 be the cyclic group of order 2, generated by y.

Let δ be the only non-trivial homomorphism from S_3 to C_8. Let W denote the group $S_3 \times C_8$. Let α be the automorphism of W given by $\alpha(\lambda, x^i) = (\lambda, x^i\delta(\lambda))$, and let β be the automorphism of W given by $\beta(\lambda, x^i) = (\lambda, x^{5i}\delta(\lambda))$.

Since α has order two, we can define the group G as the semidirect product of W with C_2 by α, that is, in G we have that $y(\lambda, x^i)y = \alpha(\lambda, x^i)$. Similarly, we define the group Q as the semidirect product of W and C_2 by β; in Q we have that $y(\lambda, x^i)y = \beta(\lambda, x^i)$. We shall denote the elements of both G and Q as $\lambda x^i y^j$.

Note that in G, x and y commute, and the centre of G is therefore the subgroup generated by x, which is a subgroup of order 8; however, x and y do not commute in Q, and the centre of Q is the subgroup generated by x^2, which is a subgroup of order 4. In particular, we also have that G and Q are non-isomorphic groups of order 96.

The following theorem can be found in [4].

Theorem 2. *Let S be a subset of G (and therefore S is also a subset of Q). Then S is a subgroup of G if and only if S is a subgroup of Q. Moreover, two subgroups are conjugate in G if and only if they are conjugate in Q, and the identity map on the family of conjugacy classes of subgroups defines and isomorphism between the tables of marks of G and Q.*

We use this fact to prove our main result.

Theorem 3. *The subgroup of Q generated by x is not a characteristic subgroup. In particular, the isomorphism of tables of marks between G and Q maps the centre of G to a non-characteristic subgroup of Q.*

Proof. We construct an automorphism of Q that does not preserve the subgroup generated by x. Let $\eta : Q \longrightarrow Q$ be given by

$$\eta(\lambda x^i y^j) = \lambda x^{3i+6i^2+\big(1-Sgn(\lambda)\big)(2i+3)} y^{i+j+\frac{1-Sgn(\lambda)}{2}}$$

We claim that for a generator g of Q and an arbitrary $\lambda x^i y^j$ we have that $\eta(g\lambda x^i y^j) = \eta(g)\eta(\lambda x^i y^j)$, where g can be $(1,2), (1,2,3), x, y$, so η is indeed a homomorphism.

$g = (1,2)$:

$$\begin{aligned}
&\eta\big((1,2)(\lambda x^i y^j)\big) = \eta\big((1,2)\lambda x^i y^j\big) \\
&= (1,2)\lambda x^{5i+6i^2-2iSgn\big((1,2)\lambda\big)-3sgn\big((1,2)\lambda\big)+3} \\
&y^{i+j+\frac{1-Sgn\big((1,2)\lambda\big)}{2}} \\
&= (1,2)\lambda x^{5i+6i^2+2iSgn(\lambda)+3sgn(\lambda)+3} y^{i+j+\frac{1+Sgn(\lambda)}{2}}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\eta\big((1,2)\big)\eta\big(\lambda x^i y^j\big) \\
&= \big((1,2)x^6 y\big)\big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3} \\
&y^{i+j+\frac{1-Sgn(\lambda)}{2}}\big) \\
&= (1,2)\lambda \\
&x^{6+5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2(1-Sgn(\lambda))} \\
&y^{1+i+j+\frac{1-Sgn(\lambda)}{2}} \\
&= (1,2)\lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+7} y^{1+i+j\frac{1-Sgn(\lambda)}{2}}
\end{aligned}$$

These two expressions coincide, because:

$$\begin{aligned}
&\big(5i + 2iSgn(\lambda) + 3sgn(\lambda) + 3\big) - \\
&\big(i - 2iSgn(\lambda) - Sgn(\lambda) + 7\big) \\
&= 4i + 4isgn(\lambda) + 4sgn(\lambda) + 4 \\
&= 4\big((1 + Sgn(\lambda))(i + 1)\big)
\end{aligned}$$

$g=(1,2,3)$:

$$\eta\big((1,2,3)(\lambda x^i y^j)\big)=\eta\big((1,2,3)\lambda x^i y^j\big)$$
$$=(1,2,3)\lambda x^{5i+6i^2-2iSgn\big((1,2,3)\lambda\big)-3Sgn\big((1,2,3)\lambda\big)+3}$$
$$y^{i+j+\frac{1-Sgn\big((1,2,3)\lambda\big)}{2}}$$
$$=(1,2,3)\lambda x^{5i+6i^2-2iSgn(\lambda)-3sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}$$

On the other hand:

$$\eta\big((1,2,3)\big)\eta\big(\lambda x^i y^j\big)$$
$$=\big((1,2,3)\big)\big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$
$$y^{i+j+\frac{1-Sgn(\lambda)}{2}}\big)$$
$$=(1,2,3)\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$
$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

$g=y$:

$$\eta(y\lambda x^i y^j)=\eta(\lambda x^{5i+2-2Sgn(\lambda)}y^{1+j})$$
$$=\lambda x^{5[5i+2-2Sgn(\lambda)]+6[5i+2-2Sgn(\lambda)]^2-}$$
$$^{2[5i+2-2Sgn(\lambda)]Sgn(\lambda)-3Sgn(\lambda)+3}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$
$$=\lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+1}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

On the other hand:

$$\eta(y)\eta(\lambda x^i y^j)=y\big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}$$
$$y^{i+j+\frac{1-Sgn(\lambda)}{2}}\big)$$
$$=\lambda x^{5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2[1-Sgn(\lambda)]}$$
$$y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$
$$=\lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+1}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}$$

$g = x$

$$\begin{aligned}
&\eta(x\lambda x^i y^j) = \eta(\lambda x^{1+i} y^j)\\
&= \lambda x^{5(1+i)+6(1+i)^2-2(1+i)Sgn(\lambda)-3Sgn(\lambda)+3}\\
&y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\
&= \lambda x^{i+6i^2-2iSgn(\lambda)-5Sgn(\lambda)+6} y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\eta(x)\eta(\lambda x^i y^j) = (xy)\big(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}\\
&y^{i+j+\frac{1-Sgn(\lambda)}{2}}\big)\\
&= \lambda x^{1+5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2(1-Sgn(\lambda))}\\
&y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\
&= \lambda x^{2+i+6i^2-2iSgn(\lambda)-Sgn(\lambda)} y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}
\end{aligned}$$

These two expressions coincide because:

$$\big(6 - 5Sgn(\lambda)\big) - \big(2 - Sgn(\lambda)\big) = 4(1 - Sgn(\lambda))$$

Therefore η is a group homomorphism.
Moreover,

$$(1,2) = \eta((1,2)x^6 y), \qquad (1,2,3) = \eta(1,2,3),$$
$$x = \eta(xy), \qquad y = \eta(y)$$

so η must be an automorphism. Finally, note that $\eta(x) = xy$, so the subgroup generated by x is not a characteristic subgroup of Q.

□

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