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TRIANGULAR PARTITIONS WITH AUGMENTED FIRST ROWS AND WEIGHTS FOR THE SYMMETRIC GROUPS IN CHARACTERISTIC TWO

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ABSTRACT. Alperin's weight conjecture for the symmetric groups has been proved using an enumeration of the weights and the simple modules (see [2]), but so far there is no explicit way to associate weights with simple modules. In this paper we prove that some weights for the symmetric groups in characteristic two can be found inside the Brauer quotients of the simple modules parameterized by partitions consisting of a triangle with an enlarged first row. Furthermore, we find subgroups of S_n which are minimal such that their Brauer quotients have a simple projective summand.

1. Introduction

One of the most important (and difficult) open problems in the representation theory of finite groups is Alperin's weight conjecture. Even though this conjecture has already been established for several families of groups (see Section 3), some of these proofs are just enumerations of the sets of weights and irreducibles, with no explicit correspondence between them. Such is the case of the symmetric groups.

Alperin and Fong proved in [2] that Alperin's conjecture holds for the symmetric groups, so we know that the number of weights for kS_n equals the number of simple kS_n-modules, where k is a field of characteristic $p > 0$. In [11] we proved that three infinite families of 2-regular partitions - which, as is well-known, parameterize the irreducible kS_n-modules in characteristic two - have Brauer quotients which are simple and projective. Hence these Brauer quotients represent weights for kS_n, and at least for the simple modules parameterized by these special partitions we have a way of assigning explicit weights.

In this paper we prove that for another family of partitions, the Brauer quotients of the simple modules parameterized by the partitions in the family have simple projective summands. Although we cannot guarantee that their Brauer quotients are themselves simple (and hence projective), the existence of a simple projective summand is enough to once again assign specific weights to the simple modules parameterized by the partitions in our family.

We shall work with an algebraically closed field k of characteristic two. The partitions we shall consider are the ones that can be obtained by adding n nodes to the first row of a triangular partition of size t. Since triangular partitions

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are the only two-cores, they are the ones that correspond to the blocks of defect zero of the symmetric groups in characteristic two.

On the other hand, to each triangular partition was appended a horizontal partition, which parameterizes the trivial representation of kS_n. The most natural way to assign a weight to the trivial kS_n-module in general is to choose (P, k), where P is a Sylow p-subgroup of S_n and k is the trivial $N_{S_n}(P)/P$-module. It seems reasonable to expect that a partition that can be formed from a triangle and a horizontal line should correspond to a weight that encodes this information. In fact, the weight that we assign to this partition - that is, the weight that is a summand of the Brauer quotient - consists of a Sylow 2-subgroup of S_n and the only simple projective module for kS_t. This weight encapsulates the essential information about the partition.

In Section 2 we give James' construction of the irreducible kS_n-modules in characteristic p. In Section 3 we define weights and state Alperin's conjecture. In Section 4 we define Brauer quotients. As a matter of fact, we shall only work with the "fixed points Brauer quotients", a special case of a general construction that can be used to study Mackey Functors (see [10]).

In Section 5 we prove out main result. We note that when a weight subgroup of S_m has fixed points on $M = \{1, \dots, m\}$, then the weight breaks down as a simple projective module for the symmetric group of the fixed points on M and a weight for the symmetric group of the complement of the fixed points. We use this idea to chase the simple projective module we want through the whole process until we reach the Brauer quotient. In other words, if μ denotes the triangular partition of size t and λ the partition obtained by adding n extra nodes to the first row of μ, then we prove that the simple projective kS_t-module D^μ is a direct summand of M^λ, S^λ, D^λ, $(D^\lambda)^H$ and of the Brauer quotient of D^λ with respect to H, where H is a Sylow 2-subgroup of S_n. We also prove that the subgroup H is minimal with this property, that is, if K is a proper subgroup of H, then the Brauer quotient of D^λ with respect to K does not have any simple projective direct summands.

2. Some important kS_n-modules

We define the modules M^λ, S^λ and D^λ following James [5]. The simple kS_n-modules, as is well known, can be parameterized by certain partitions of n called p-regular, where p is the characteristic of the field k. Moreover, it is possible to construct each simple module from its associated partition. In this section n is a natural number, k is a field of characteristic $p > 0$ and λ is a partition of n.

Definition (2.1). A *λ-tableau* is one of the $n!$ arrays of integers obtained by replacing each node in the partition λ by one of the integers $1, 2, \dots, n$, allowing no repeats. If t is a tableau, its *row stabilizer*, R_t, is the subgroup of S_n consisting of the elements which fix all rows of t setwise. The *column stabilizer* of t, denoted C_t, is the subgroup of S_n consisting of the elements which fix all columns of t setwise. The *signed column sum* of t, denoted κ_t, is the element of kS_n given by

$$\kappa_t := \sum_{\pi \in C_t} (-1)^{\mathrm{sign}(\pi)} \pi.$$

We define an equivalence relation on the set of λ-tableaux by $t_1 \sim t_2$ if and only if $\pi t_1 = t_2$ for some $\pi \in R_{t_1}$. The *tabloid*, $\{t\}$ containing t is the equivalence class of t under this relation. The kS_n-module $M^\lambda = M_k^\lambda$ is the vector space over k whose basis elements are the various λ-tabloids. The *polytabloid*, e_t, associated with the tableau t is given by

$$e_t := \kappa_t \{t\}.$$

The *Specht module*, $S^\lambda = S_k^\lambda$ for the partition λ is the submodule of M^λ spanned by polytabloids (this is indeed a kS_n-module).

We also define an S_n-invariant, symmetric, non-singular *bilinear form* $<,>$ on M^λ, whose values on pairs of tabloids is given by

$$< t_1, t_2 > := \begin{cases} 1 & \text{if } t_1 = t_2, \\ 0 & \text{if } t_1 \neq t_2. \end{cases}$$

The partition λ is *p-singular* if it has at least p rows of the same size; otherwise, λ is *p-regular*. The module $D^\lambda = D_k^\lambda$ is defined as

$$D^\lambda := S^\lambda / (S^\lambda \cap S^{\lambda \perp})$$

where λ is a p-regular partition.

THEOREM (2.2). (James) *As λ varies over p-regular partitions of n, D^λ varies over a complete set of inequivalent irreducible kS_n-modules. Each D^λ is self-dual and absolutely irreducible. Every field is a splitting field for S_n.*

For a proof of this result, see [5].

Example (2.3). Let $\lambda = (n)$ be the partition with just one row of length n. Then all λ-tableaux are row equivalent, so there is only one λ-tabloid, and $M^{(n)} = k$ is the one dimensional trivial module. We also have that $S^{(n)} = D^{(n)} = k$.

3. Alperin's Conjecture

We give the definition of weight and formulate Alperin's Conjecture in its most general form. We mention some classes of groups for which it is known to be valid (including the symmetric groups) and we note the possible advantages of a combinatorial proof, that is, an explicit bijection between weights and irreducible modules.

Throughout this section, G will be a finite group, p a prime number, and k a splitting field for G in characteristic p. All our modules will be finite dimensional over k.

Definition (3.1). A *weight* for G is a pair (Q, S) where Q is a p-subgroup and S is a simple module for $k[N_G(Q)]$ which is projective when regarded as a module for $k[N_G(Q)/Q]$.

Remark (3.2). Since S is $k[N(Q)]$-simple and Q is a p-subgroup of $N_G(Q)$, it follows that Q acts trivially on S, so S is also a $k[N_G(Q)/Q]$-module and the definition makes sense. Moreover, S is $k[N_G(Q)/Q]$-simple as well.

Remark (3.3). If we replace S by an isomorphic $k[N_G(Q)]$-module we consider this the same weight, and we make the same identification when we replace Q by a conjugate subgroup (so that the normalizers will be conjugate, too).

Now we can formulate the main problem that we shall discuss in this section.

THEOREM (3.4). (*Alperin's Conjecture*) *The number of weights for G equals the number of simple kG-modules.*

A stronger version of the preceding statement is that there is a bijection within each block of the group algebra.

Definition (3.5). If (Q, S) is a weight for G, then S belongs to a block b of $N_G(Q)$ and this block corresponds with a block B of G via the Brauer correspondence; hence we can say that the weight (Q, S) *belongs* to the block B of G so the weights are partitioned into blocks.

THEOREM (3.6). (Alperin's Conjecture, Block Form) *The number of weights in a block of G equals the number of simple modules in the block.*

This version of the conjecture implies the original one, as it can be obtained by summing the equalities from the stronger conjecture over the blocks. This stronger conjecture has been proved when G is a:

- Finite group of Lie type and characteristic p (Cabanes, [3]).
- Solvable group (Okuyama, [8]).
- Symmetric group (Alperin and Fong, [2]).
- $GL(n, q)$ and p does not divide q (Alperin and Fong, [2]).

Alperin and Fong's proof in the case of symmetric groups was just an observation of a numerical equality which did not suggest a deeper reason for the relationship. For finite groups in general one does not expect to have any canonical bijection between weights and simple modules; as a matter of fact, Alperin himself says this is unlikely (see [1], p 369). For groups of Lie type in their defining characteristic there is a canonical bijection (described in [1]). Since symmetric groups and groups of Lie type have such strong connections in their representation theory, it is reasonable to ask whether there is some canonical bijection in the case of symmetric groups.

If true, Alperin's conjecture would imply a number of known results, until now unrelated (see [1]). It is also reasonable to expect that if an explicit bijection can be given to prove it, this may reveal new connections between simple kG-modules and weights; there are many results known about the former, and the latter are related to the blocks of defect zero, which are not as easy to deal with as the simple modules. In fact, this is really the true importance of Alperin's conjecture in that it provides a connection between the blocks of defect zero and the set of all simple modules. More specifically, Alperin's conjecture has been shown by Robinson [7] to be equivalent to a statement which expresses the number of blocks of defect zero of a group in terms of the number of p-modular irreducibles of sections of the group of the form $N_G(P)/P$, $P \leq G$ a p-subgroup. These latter numbers are easy to compute, since by a theorem of Brauer the number of p-modular irreducibles of a group equals the number of p-regular conjugacy classes.

4. Brauer quotients

We define Brauer quotients, a relatively new tool in the representation theory of groups. In this section k is an arbitrary field, G an arbitrary finite group, H a subgroup of G, and V a kG-module. We denote by V^G the fixed points of V under G.

Definition (4.1). The map $tr_H^G{:}V^H \longrightarrow V^G$ given by

$$m \mapsto \left(\sum_{i=1}^{l} g_i\right) m,$$

where $G = \sqcup_{i=1}^{l} g_i H$, is called the *relative trace* from H to G. The *Brauer quotient* of V with respect to H is defined as

$$\overline{FP}_V(H) := V^H / \sum_{K<H} tr_K^H(V^K).$$

This is a $k[N_G(H)]$-module, where H acts trivially, so it is a $k[N_G(H)/H]$-module. The preceding definition is a particular example of the Brauer quotient of a Mackey functor: in our case we are using the fixed points Mackey functor. Constructions such as this appear in recent work by various authors such as Puig and Thévenaz, see [9].

5. Augmented triangular partitions and weights

In this section we prove our main result. Unless otherwise stated, k is an algebraically closed field of characteristic two, t is a triangular number, μ is the triangular partition of size t, n a natural number, and λ the partition of $n+t$ obtained by adjoining n nodes to the first row of μ.

First we recall the fundamental property of triangular partitions. For a proof of this result, see [6].

LEMMA (5.1). *We have that $S^\mu = D^\mu$ is simple and projective as a kS_t-module. Furthermore, kS_n has a simple projective module if and only if n is a triangular number, and in this case there is only one simple projective module.*

We remind the reader of some useful facts about the representations of Cartesian products of groups.

PROPOSITION (5.2). *Let k be a splitting field for the finite groups R and S, let U, T be finite dimensional modules for kR and kS respectively, and let $k(R\times S)$ act on $U \otimes_k T$ via $(r,s)(u\otimes t) = ru \otimes st$. Then*

(i) $U\otimes_k T$ is a simple $k[R\times S]$-module if and only if U is a simple kR-module and T is a simple kS-module.

(ii) $U \otimes_k T$ is a projective $k[R\times S]$-module if and only if U is a projective kR-module and T is a projective kS-module.

Proof. (i) A proof of this result can be found in [4], Theorem (10.33).

(ii) It is clear that the tensor product of two projective modules is projective. Assume that $U \otimes T$ is a projective module for the group $R \times S$. Then its restriction to the subgroup $R = R \times \{1\}$ is projective, and this is isomorphic to several copies of the R-module U (as many copies as the dimension of T over k), which proves that U must be projective. A similar argument proves that T is a projective kS-module. □

This situation arises naturally when a subgroup H of S_n has fixed points on the set $\{1, \ldots, n\}$.

LEMMA (5.3). *Let H be a subgroup of the symmetric group S_M with fixed points F and let Θ be the complement of F in M. Then*

$$N_{S_M}(H)/H = \big(N_{S_\Theta}(H)/H\big) \times S_F$$

Proof. One containment is immediate. Now let $\tau \in N_{S_M}(H)$. Then τ permutes the fixed points F of H, so $\tau = \alpha\beta$ with $\alpha \in S_\Theta$ and $\beta \in S_F$. It follows that α is in $N_{S_\Theta}(H)$. □

As a result, a module for the quotient group $N_{S_M}(H)/H$ is really a module for the product $\big(N_{S_\Theta}(H)/H\big) \times S_F$. In this context, if we refer to a kS_F-module U as a $k[N_{S_M}(H)/H]$-module, we mean $k \otimes_k U$, that is, $N_{S_\Theta}(H)/H$ acts trivially on U. Notice that the $k[N_{S_M}(H)/H]$-module $k \otimes_k U$ will be simple and projective if and only if both k and U are simple and projective modules for $N_{S_\Theta}(H)/H$ and S_F respectively, that is, if and only if H is a Sylow 2-subgroup of S_Θ and U is a simple and projective kS_F-module.

Combining the previous results we have the following.

COROLLARY (5.4). *Let H be a weight subgroup of S_n, with no fixed points on $\{1, \ldots, n\}$. For each $q > n$, regard S_n as the subgroup of S_q that fixes the points $x > n$. Then H is a weight subgroup of S_q if and only if $q - n$ is a triangular number. In this case, the number of isomorphism classes of simple projective $N_{S_n}(H)/H$-modules equals the number of isomorphism classes of simple projective $N_{S_q}(H)/H$-modules.*

Proof. The group $N_{S_q}(H)/H$ is isomorphic to $\big(N_{S_n}(H)/H\big) \times S_{q-n}$, which has a simple projective module if and only if $q - n$ is a triangular number. The last part follows from the fact that the simple projective modules for a product of groups are the tensor products of their respective simple projective modules. □

We must prove that there is a simple projective kS_t-module (the one parameterized by μ) inside M^λ, S^λ and D^λ. In order to study the restriction of the kS_{n+t}-module M^λ to kS_t, we must analyse some special subpartitions that can be obtained from λ.

LEMMA (5.5). *Let ν be any partition, with rows $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_s$. Remove n symbols and push up, obtaining a composition, $\hat{\nu}$ (the rows of $\hat{\nu}$ need not be in descending order). Rearrange the rows of $\hat{\nu}$ to get a partition $\tilde{\nu}$. Then $\tilde{\nu}$ "fits inside" ν, that is, $\tilde{\nu}_i \leq \nu_i$ for all i.*

Proof. Induction on n. Let $n = 1$, and let j be the index of the row where a node is deleted. Then the j-th row of the composition $\hat{\nu}$ is $\nu_j - 1$, but all the other rows coincide with those of ν. Let a be the smallest number such that $\nu_j > \nu_a$ (possible, since $\nu_j > 0$). Then we have $\nu_j = \nu_{j+1} = \cdots = \nu_{a-1} > \nu_a$, so $\nu_j - 1 \geq \nu_a$, and rearranging the rows of $\hat{\nu}$ in decreasing order we get that $\tilde{\nu}_i = \nu_i$ if $i \neq a-1$, and $\tilde{\nu}_{a-1} = \nu_{a-1} - 1$. Thus $\tilde{\nu}$ fits inside ν.

Now assume the result holds for n symbols. Removing n symbols from ν gives a partition ψ that fits inside ν, and removing one symbol from ψ gives a partition ϕ that fits inside ψ, so ϕ fits inside ν as well. □

Remark (5.6). When we remove one node, it suffices to take it from a row that is strictly larger than the next (no shifting will occur). Note also that both $\hat{\nu}$ and $\tilde{\nu}$ depend on the choice of the rows from which the n symbols are removed.

LEMMA (5.7). *Let $\tilde{\lambda}$ be any partition obtained from λ by removing n nodes as above. Then $\tilde{\lambda}$ dominates μ, and $\tilde{\lambda} = \mu$ if and only if the n nodes were removed from the first row.*

Proof. To show $\sum_{i=1}^{r} \tilde{\lambda}_i \geq \sum_{i=1}^{r} \mu_i$ for all $r \geq 1$, we show that $\sum_{i>r} \tilde{\lambda}_i \leq \sum_{i>r} \mu_i$, and this follows from the previous lemma. The only way to obtain μ is to remove n nodes from the first row. □

PROPOSITION (5.8). *The module M^λ restricted to S_t is a direct sum of M^ν, where each ν is obtained from λ as $\tilde{\lambda}$, removing the nodes with the symbols $t+1, \ldots, t+n$ from each λ-tabloid. The module M^μ occurs precisely once.*

Proof. The module M^λ is the permutation module with λ-tabloids as a basis. Restricting this to S_t we get again a permutation module. Let α be a λ-tabloid. We claim that the orbit of α under S_t gives rise to a permutation module $M^{\tilde{\lambda}}$, where $\tilde{\lambda}$ is obtained by removing n nodes from λ as described in Lemma (5.5), the n nodes being deleted from the rows where the numbers $t+1, \ldots, t+n$ are placed in the λ-tabloid α. These two kS_t-permutation modules are isomorphic because their underlying S_t-sets are isomorphic, which in turn follows from the fact that the stabilizer in S_t of α coincides with the stabilizer of the $\tilde{\lambda}$-tabloid obtained from α by deleting the n nodes from the rows where the symbols $t+1, \ldots, t+n$ are. The only time that we get M^μ is when we remove all n nodes from the first row, which can be done in exactly one way, so M^μ appears exactly once as a summand (α is a tabloid, so the order of the symbols within each row is irrelevant). □

COROLLARY (5.9). *The module M^λ restricted to S_t has exactly one composition factor isomorphic to D^μ.*

Proof. It follows from the previous proposition and the fact (see [5]) that the composition factors of M^ν are all of the form D^α, where α dominates ν (and, when ν is 2-regular, exactly one composition factor is isomorphic to D^ν). □

The following result will prove that, as a kS_t-module, D^μ is a direct summand of S^λ, D^λ and even $(D^\lambda)^K$ for any subgroup K of S_n.

PROPOSITION (5.10). *Let $\varphi = \varphi_{t,n} : M^\mu \longrightarrow M^\lambda$ be the map given by adjoining the numbers $t+1, t+2, \ldots, t+n$ to the first row of each μ-tabloid (and extending by linearity). Then φ is a monomorphism of kS_t-modules. Moreover, φ sends S^μ into S^λ, and if $\pi : S^\lambda \longrightarrow D^\lambda$ is the natural quotient map, then $\pi \circ \varphi$ sends S^μ isomorphically into a summand (as a kS_t-module) of the fixed points of D^λ under the subgroup $S_{\{t+1,t+2,\ldots,t+n\}}$. In particular, if K is any subgroup of S_n, there is a monomorphism of S_t- modules $D^\mu \longrightarrow (D^\lambda)^K$.*

Proof. We see that φ is a monomorphism of S_t-modules. Let α be a μ-tableau, and let β be the tableau obtained from α by adjoining the numbers $t+1, t+2, \ldots, t+n$ to the first row. Then the column stabilizer in S_t of α is the same as the column stabilizer of β in S_{t+n}, so the polytabloid generated by α will be sent by φ to the polytabloid generated by β. Since this holds for any tableau α, we have that φ sends S^μ into S^λ. The map φ also preserves the standard bilinear form on M^μ, and since $S^\mu = D^\mu$, we have that $\varphi(S^\mu) \cap (S^\lambda)^\perp$ is contained in $\varphi(S^\mu) \cap (\varphi(S^\mu))^\perp = \varphi(S^\mu \cap (S^\mu)^\perp) = 0$, so the composition $\pi \circ \varphi$ is injective. The rest follows from the fact that S^μ is S_t-projective. □

COROLLARY (5.11). *The module S^λ restricted to S_t has exactly one composition factor isomorphic to D^μ, and so do D^λ and $(D^\lambda)^K$ for any subgroup K of S_n.*

Proof. The module $S^\lambda \downarrow_{S_t}$ is a submodule of $M^\lambda \downarrow_{S_t}$, so it has at most one D^μ as composition factor (see Corollary (5.9)). But $D^\lambda \downarrow_{S_t}$ is a quotient module of $S^\lambda \downarrow_{S_t}$, and $(D^\lambda)^K$ is a submodule of $D^\lambda \downarrow_{S_t}$, so these also have at most one D^μ as a composition factor. Finally, Proposition (5.10) states that $(D^\lambda)^K$ has at least one composition factor isomorphic to D^μ, hence so do $D^\lambda \downarrow_{S_t}$ and $S^\lambda \downarrow_{S_t}$. □

COROLLARY (5.12). *Let H be a 2-subgroup of S_n, and K a proper subgroup of H. Let $\psi : D^\mu \longrightarrow (D^\lambda)^K$ be the monomorphism of kS_t-modules defined in Proposition (5.10). Then $tr_K^H \circ \psi = 0$.*

Proof. We have that $\psi(D^\mu)$ is contained in $(D^\lambda)^H$, so $tr_K^H(\psi(v)) = [H : K]\psi(v) = 0$. □

We analyse what the relative traces do to this copy of D^μ in the fixed points $(D^\lambda)^H$.

LEMMA (5.13). *If $K \leq H \leq S_n$ then*

$$tr_K^H : (D^\lambda)^K \longrightarrow (D^\lambda)^H$$

is a morphism of kS_t-modules, where S_t is regarded as a subgroup of the quotient $N_{S_{t+n}}(H)/H$.

Proof. Let $y \in S_t$. Then for any $x \in H$ we have $yx = xy$, so $ytr_K^H(v) = y\sum x_i v = \sum yx_i v = \sum x_i(yv) = tr_K^H(yv)$, where $yv \in (D^\lambda)^K$. □

COROLLARY (5.14). *If $K \leq H \leq S_n$ then $Im(tr_K^H)$ is an S_t-submodule of $(D^\lambda)^H$.*

LEMMA (5.15). *Let $V_1, \ldots, V_r$ be a family of submodules of a module V. Then $\sum V_i$ is a quotient module of $\oplus V_i$.*

Proof. The map $(v_1, \dots, v_r) \mapsto v_1 + \cdots + v_r$ is surjective. $\square$

COROLLARY (5.16). *Let H be a 2-subgroup of S_n, and K a proper subgroup of H. Then*

(i) The kernel of $tr_K^H : (D^\lambda)^K \longrightarrow (D^\lambda)^H$ has D^μ as a composition factor.

(ii) The image of tr_K^H does not have D^μ as a composition factor.

(iii) No composition factor of $\sum_{K<H} Im(tr_K^H)$ is isomorphic to D^μ.

(iv) As a kS_t-module (and not necessarily as a $k[N_{S_{n+t}}(H)/H]$-module), the Brauer quotient $\overline{FP}_{D^\lambda}(H)$ has D^μ as a composition factor of multiplicity one.

Proof. (i) follows from Corollary (5.12); (ii) follows from Corollary (5.11) and (i); (iii) follows from (ii) and Lemma (5.15); (iv) follows from (iii) and Corollary (5.11). $\square$

Notice the remark in part (iv). Note also that if $H \le S_n$, then $N_{S_{n+t}}(H)/H \cong S_t$ if and only if H is a Sylow 2-subgroup of S_n.

We are almost done. In order to prove the minimality condition, we prove that there are no simple projective kS_d-modules arising from a triangular number d greater than t.

LEMMA (5.17). *Let λ be a partition of m, μ a 2-regular partition of t, $t < m$. If D^μ is a composition factor of $M^\lambda \downarrow_{S_t}$, then there exists a partition ν of t such that ν fits inside λ and μ dominates ν.*

Proof. By Proposition (5.8), $M^\lambda \downarrow_{S_t}$ is a direct sum of M^ν, where each ν fits inside α. If one of the M^ν has D^μ as a composition factor, we know (see [5]) that μ dominates ν. $\square$

LEMMA (5.18). *Let t be a triangular number, μ the triangular partition of size t, λ the partition obtained by adding n nodes to the first row of μ. Let ν be a subpartition of λ (so $\mu_i \ge \nu_i$ for all $i \ge 2$) and α a triangular partition with $|\alpha| = |\nu|$ and that dominates ν. Then $\mu_1 \ge \alpha_1$, and hence $|\mu| \ge |\alpha|$.*

Proof. Suppose $\alpha_1 > \mu_1$. Then α is a larger triangle than μ, so there exists r such that $\alpha_r > 0 = \mu_r$, so $|\alpha| \ge \sum_{i=1}^r \alpha_i > \alpha_1 + \sum_{i=2}^r \mu_i \ge \nu_1 + \sum_{i=2}^r \nu_i = |\nu|$, contradicting the fact that α and ν had the same size. $\square$

We can prove our main result.

THEOREM (5.19). *Let k be an algebraically closed field of characteristic two, t a triangular number, μ the triangular partition of size t, n a natural number, and λ the partition of $n+t$ obtained by adjoining n nodes to the first row of μ. Let H be a Sylow 2-subgroup of S_n. Then $N_{S_{n+t}}(H)/H \cong S_t$, and $\overline{FP}_{D^\lambda}(H)$ has a simple projective summand of multiplicity one, given by D^μ. If K is a proper subgroup of H, then $\overline{FP}_{D^\lambda}(K)$ contains no simple projective summands.*

Proof. The first part follows from Corollary (5.16). Let r be the number of points in $\{1, \dots, n\}$ that K moves. We have that $N_{S_{n+t}}(K) = S_{n+t-r} \times N_{S_r}(K)$, and $N_{S_{n+t}}(K)/K \cong S_{n+t-r} \times N_{S_r}(K)/K$. Suppose that $\overline{FP}_{D^\lambda}(K)$ has a simple projective summand. Then $n+t-r$ is a triangular number, and its triangular partition α is such that D^α is a composition factor of $M^\lambda \downarrow_{S_{|\alpha|}}$. By Lemma (5.17), there exists a subpartition ν of λ such that $|\alpha| = |\nu|$ and α dominates ν. By Lemma (5.18), $|\alpha| \leq |\mu|$, i.e. $n + t - r \leq t$, so $n = r$.

We have then that $N_{S_{n+t}}(K) = S_t \times N_{S_n}(K)$, so $N_{S_{n+t}}(K)/K \cong S_t \times N_{S_n}(K)/K$. A simple projective module for the latter group must be of the form $D^\mu \otimes B$, where B is a simple projective $N_{S_n}(K)/K$-module and D^μ is the only simple projective S_t-module. If $\overline{FP}_{D^\lambda}(K)$ had such a summand, then restricting to S_t the Brauer quotient would have a summand $(D^\mu \otimes B) \downarrow_{S_t} \cong (D^\mu)^{\dim(B)}$, so D^μ would have multiplicity $\dim(B)$, and since D^μ appears only once, B must have dimension 1. Thus B cannot be projective for $N_{S_n}(K)/K$ if K is a proper subgroup of H since then $N_{S_n}(K)/K$ has order divisible by 2. $\square$

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