SOME SIMPLE PROJECTIVE BRAUER QUOTIENTS OF SIMPLE MODULES FOR THE SYMMETRIC GROUPS IN CHARACTERISTIC TWO

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Abstract. By Alperin's weight conjecture [3] the number of simple kS_n-modules equals the number of weights for S_n, where S_n is the symmetric group on n symbols and k is a field of characteristic $p > 0$. In this paper we answer the question "when is the Brauer quotient of a simple F_2S_n-module V with respect to a subgroup H of S_n both simple and projective as an $N_{S_n}(H)/H$-module?", in some special cases. Remarkably, in each case there is only one such subgroup H (up to conjugacy).

1. Introduction

Alperin's weight conjecture remains one of the most important open problems in the representation theory of finite groups. This conjecture has already been established for several families of groups (see section 2.1). A natural question to ask is whether for these families one can establish an explicit bijection between weights and simple modules. Alperin provided (in [2]) an explicit bijection between weights and simple modules using Green correspondents and several facts from the representation theory of groups of Lie type, but his proof cannot be adapted to other finite groups.

What we are looking for is a criterion which, given a weight (Q, S) and a simple module V for a finite group G, will determine if they correspond. One might expect the $N_G(Q)/Q$-module S to appear inside V^Q (the fixed points of V under Q) in some way, since Q acts trivially on S. The following result seems encouraging:

Proposition 1.1. *Let G be any finite group, k any field of characteristic $p > 0$, (Q, S) a weight for G. Then there is a simple kG-module V such that S is a direct summand (or equivalently, a composition factor) of V^Q as $N_G(Q)/Q$-modules.*

Proof. Let M be the regular kG-module. Since M^Q contains a copy of the regular $k[N_G(Q)/Q]$-module, S is a direct summand of M^Q. If W is a kG-submodule of M and W^Q does not have S as a composition factor, then S must be a composition factor of M^Q/W^Q, which is a submodule of $(M/W)^Q$. Moving down along a composition series of M will thus yield a simple kG-module V whose fixed points under Q contain a copy of S. □

If such simple module were uniquely determined, and if every simple module could be linked to exactly one weight in such manner, we would have an explicit bijection. However, the fixed points are too large, in the sense that one weight

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can be embedded in the fixed points of several simple kG-modules. Indeed, the weight (P, k) consisting of a Sylow p-subgroup and the trivial module appears in the fixed points of every simple kG-module! If one were able to find a suitable $N_G(Q)/Q$-module, smaller than V^Q but large enough to contain S, then it might be possible to prove Alperin's conjecture by means of an explicit bijection, making (Q, S) correspond to V. Unfortunately it is not easy to tell exactly how small that module ought to be, or even if such a procedure is possible for all finite groups.

In this paper we prove that for several families of simple modules V for the symmetric group S_n over the field F_2, there is exactly one weight (Q, S) such that the Brauer quotient (defined in section 2.2) of V with respect to Q is isomorphic to S as $N_G(Q)/Q$-modules. That is, if instead of using fixed points one were to use Brauer quotients, there are infinitely many instances of simple modules for the symmetric groups to which one and only one weight can be assigned.

In order to compute these Brauer quotients, we use a characterization of some of the simple modules for S_n in terms of subsets of $\{1, \ldots, n\}$ (case $V = D^{(n-1,1)}, n$ odd), equivalence classes of such subsets under complements (case $V = D^{(n-1,1)}, n \equiv 2(\bmod 4)$), and in terms of formal linear combinations of subsets of $\{1, \ldots, n\}$ of size two (case $V = D^{(n-2,2)}, n \equiv 3(\bmod 4)$). We prove that for a simple module V in each of the previous cases, there is exactly one subgroup H of S_n (up to conjugacy) such that the Brauer quotient of V with respect to H is simple and projective.

Our approach is very computational.That is, we often find explicitly the fixed points and the images of the relative trace maps from maximal subgroups. These computations will be difficult to carry out for more complicated simple modules, but using computer software written in GAP ([6]) in collaboration with Peter Webb, we have been able to establish experimentally that many other simple modules for S_n have simple projective Brauer quotients. These computations and some connections we have discovered with the combinatorics of partitions will soon appear in a separate paper.

The preliminaries have all the background material necessary to understand this paper: Alperin's conjecture, Brauer quotients, and the construction of the simple modules for the symmetric groups following James [7]. Section 3 deals with the easiest possible non-trivial case, namely, that of the irreducible module parameterized by the partition $(n-1, 1)$ where n is an odd number. The Brauer quotient of this module with respect to a subgroup of S_n is determined by the fixed points of the subgroup on the set $\{1, \ldots, n\}$. This computation is a straightforward application of the characterization of this simple module in terms of families of subsets, as described in the preliminaries.

In section 4 we analyse the Brauer quotient of the irreducible parametrized by the partition $(n-2, 2)$ where n is congruent to 3 modulo 4 and greater than 3 (the case $n = 3$ is trivial, since F_2S_3 has a block of defect zero). We no longer have a precise description of the Brauer quotient (as we did in the previous case), but we have enough information about it to determine when it will be a simple projective module. These computations are more involved, because in order to compute the Brauer quotient of a subgroup of S_n now we have to consider families of subsets of $\{1, \ldots, n\}$ of size two which are fixed under the action of the subgroup.

In section 5 we study the partition $(n-1, 1)$ when n is congruent to 2 modulo 4. In this case the Brauer quotient of a subgroup will be determined by the fixed points of the subgroup on the set $\{1, \dots, n\}$ as well as by its orbits of size 2, particularly some special orbits which we name *glued orbits*. In stark contrast with the case for the same partition with odd n, in order to have a simple projective Brauer quotient, a subgroup must not have any fixed points on the set $\{1, \dots, n\}$ (among other requirements).

In all sections, the main theorem is the last result of the section. Section 5 also has an additional result, theorem 5.16, which describes when the Brauer quotient of the Specht module is both simple and projective (in the first two cases the Spech module coincided with the simple module).

2. Preliminaries

In this section we state Alperin's conjecture and define Brauer quotients, a relatively new tool in representation theory, which has been successfully used by authors such as Puig and Thévenaz [11]. We also describe the irreducible modules of the group algebras of the symmetric groups using the characteristic-free approach in [7].

2.1. **Alperin's conjecture.** We give the definition of weight and state Alperin's conjecture in its most general form. We mention some classes of groups for which it is known to be valid (including the symmetric groups) and we note the possible advantages of a combinatorial proof, that is, an explicit bijection between weights and irreducible modules.

Throughout this section, G will be a finite group, p a prime number, and k a splitting field for G in characteristic p. All our modules will be finite dimensional over k.

Definition 2.1. A *weight* for G is a pair (Q, S) where Q is a p-subgroup and S is a simple module for $k[N_G(Q)]$ which is projective when regarded as a module for $k[N_G(Q)/Q]$.

Since S is $k[N(Q)]$-simple and Q is a p-subgroup of $N_G(Q)$, it follows that Q acts trivially on S, so S is also a $k[N_G(Q)/Q]$-module and the definition makes sense. Moreover, S is $k[N_G(Q)/Q]$-simple as well.

If we replace S by an isomorphic $k[N_G(Q)]$-module we consider this the same weight, and we make the same identification when we replace Q by a conjugate subgroup.

Alperin's conjecture: *The number of weights for G equals the number of simple kG-modules.*

A stronger version of the preceding statement is that there is a bijection within each block of the group algebra.

Definition 2.2. If (Q, S) is a weight for G, then S belongs to a block b of $N_G(Q)$ and this block corresponds with a block B of G via the Brauer correspondence; hence we can say that the weight (Q, S) *belongs* to the block B of G, so the weights are partitioned into blocks.

Alperin's conjecture, Block Form *The number of weights in a block of G equals the number of simple modules in the block.*

This version of the conjecture implies the original one, as it can be obtained by summing the equalities from the stronger conjecture over the blocks. This stronger conjecture has been proved when G is a:

- Finite group of Lie type and characteristic p (Cabanes, [4]).

- Solvable group (Okuyama, [9]).
- Symmetric group (Alperin and Fong, [3]).
- $GL(n,q)$ and p does not divide q (Alperin and Fong, [3]).

Alperin and Fong's proof in the case of symmetric groups was just an observation of a numerical equality which did not suggest a deeper reason for the relationship. For groups of Lie type in their defining characteristic there is a canonical bijection (described in [2]). Since symmetric groups and groups of Lie type have such strong connections in their representation theory, it is reasonable to ask whether there is some canonical bijection in the case of symmetric groups.

If it is true, Alperin's conjecture would imply a number of known results, until now unrelated [2]. It is also reasonable to expect that if an explicit bijection can be given to prove it, this may reveal new connections between simple kG-modules and weights; there are many results known about the former, and the latter are related to the blocks of defect zero, which are not as easy to deal with as the simple modules. In fact, this is really the true importance of Alperin's conjecture in that it provides a connection between the blocks of defect zero and the set of all simple modules. More specifically, Alperin's conjecture has been shown by Robinson [8] to be equivalent to a statement which expresses the number of blocks of defect zero of a group in terms of the number of p-modular irreducibles of sections of the group of the form $N_G(P)/P$, $P \leq G$ a p-subgroup. These latter numbers are easy to compute, since by a theorem of Brauer the number of p-modular irreducibles of a group equals the number of p-regular conjugacy classes.

2.2. **Brauer quotients.** We define Brauer quotients and state some of their properties, which we shall later use in our calculations. In this section k is an arbitrary field, F_q is the field with q elements, G an arbitrary finite group, H a subgroup of G, and V a kG-module. We denote by V^G the fixed points of V under G.

Definition 2.3. The map $tr_H^G{:}V^H \longrightarrow V^G$ given by

$$m \mapsto \left(\sum_{i=1}^{l} g_i\right) m,$$

where $G = \sqcup_{i=1}^{l} g_i H$, is called the *relative trace* from H to G. The *Brauer quotient* of V with respect to H is defined as

$$\overline{FP}_V(H) := V^H / \sum_{K<H} tr_K^H(V^K).$$

This is a $k[N_G(H)]$-module, where H acts trivially, so it is a $k[N_G(H)/H]$-module. The preceding definition is a standard example of the Brauer quotient of a Mackey functor: in our case we are using the fixed points Mackey functor. Constructions such as this appear in recent work by various authors such as Puig and Thévenaz, see [10].

Example 2.4. Let $H = \{1\}$. Then H has no proper subgroups, and so we have that $\overline{FP}_V(\{1\}) = V^H = V$ for all kG-modules V.

Example 2.5. Let k be a field of characteristic p, G any finite group, H a subgroup of G, and $V = k$ the trivial kG-module. If H is a p-subgroup of G, then (as is well known) all the relative traces from proper subgroups of H to H are identically zero, and $\overline{FP}_k(H) = k$. On the other hand, if H is not a p-subgroup, the relative trace from a Sylow p-subgroup of H to H is not zero, and $\overline{FP}_k(H) = 0$.

For convenience we state some well-known properties of Brauer quotients that will allow us to simplify our computations. The proof of the first one follows from the fact that both fixed points and relative traces preserve direct sums. The second proposition is result (27.6) in [11].

Proposition 2.6. *Brauer quotients preserve direct sums of kG-modules. That is,*

$$\overline{FP}_{V_1 \oplus V_2}(H) \cong \overline{FP}_{V_1}(H) \oplus \overline{FP}_{V_2}(H)$$

Proposition 2.7. *Let V be a kG-module that is also an H permutation module, i.e. there exists a basis X of V over k that is also an H-set, and let Y denote the fixed points of H on the set X. Then the normalizer of H in G acts on the set Y, and there is an isomorphism of $k[N_G(H)]$-modules*

$$\overline{FP}_V(H) \cong kY$$

where kY denotes the permutation module on the set Y.

We can combine the preceding results to obtain (effortlessly) the Brauer quotients of many modules.

Example 2.8. Let G be an arbitrary finite group. If $H \neq \{1\}$, then $\overline{FP}_{kG}(H) = 0$ since H does not fix any elements of G, so $\overline{FP}_{(kG)^n}(H) = 0$ and $\overline{FP}_V(H) = 0$ for any projective kG-module V. We shall later study in greater detail other connections between Brauer quotients and projectivity.

2.3. **Some important kS_n-modules.** We define the modules M^λ, S^λ and D^λ following James [7], and prove some results that we shall need later. The simple kS_n-modules, as is well known, can be parameterized by certain partitions of n called p-regular, where p is the characteristic of the field k. Moreover, it is possible to construct each simple module from its associated partition, and although in general this can be a cumbersome process, there are some partitions whose irreducible modules can be readily described using this method. Furthermore, in some cases it is also possible to find a formula that describes a simple module as a virtual difference of permutation modules, which will make the simple modules easier to manipulate. There is yet another advantage arising from the use of partitions to parameterize simple kS_n-modules: partitions are very visual objects, and it is far easier to discover a pattern by looking at a table of partitions than by studying their corresponding simple modules.

In this section n is a natural number, k is a field of characteristic $p > 0$ (unless otherwise stated) and λ is a partition of n.

Definition 2.9. A λ-*tableau* is one of the $n!$ arrays of integers obtained by replacing each node in the partition λ by one of the integers $1, 2, \ldots, n$, allowing no repeats. If t is a tableau, its *row stabilizer*, R_t, is the subgroup of S_n consisting of the elements which fix all rows of t setwise. The *column stabilizer* of t, denoted C_t, is the subgroup of S_n consisting of the elements which fix all columns of t setwise. The *signed column sum* of t, denoted κ_t, is the element of kS_n given by

$$\kappa_t := \sum_{\pi \in C_t} (-1)^{\mathrm{sign}(\pi)} \pi.$$

We define an equivalence relation on the set of λ-tableaux by $t_1 \sim t_2$ if and only if $\pi t_1 = t_2$ for some $\pi \in R_{t_1}$. The *tabloid*, $\{t\}$ containing t is the equivalence class of t under this relation. The kS_n-module $M^\lambda = M_k^\lambda$ is the vector space over k whose basis elements are the various λ-tabloids. The *polytabloid*, e_t, associated with the tableau t is given by

$$e_t := \kappa_t\{t\}.$$

The *Specht module*, $S^\lambda = S^\lambda_k$ for the partition λ is the submodule of M^λ spanned by polytabloids (this is indeed a kS_n-module).

We also define an S_n-invariant, symmetric, non-singular *bilinear form* $<,>$ on M^λ, whose values on pairs of tabloids is given by

$$< t_1, t_2 >:= \begin{cases} 1 & \text{if } t_1 = t_2, \\ 0 & \text{if } t_1 \neq t_2. \end{cases}$$

The partition λ is *p-singular* if it has at least p rows of the same size; otherwise, λ is *p-regular*. The module $D^\lambda = D^\lambda_k$ is defined as

$$D^\lambda := S^\lambda / (S^\lambda \cap S^{\lambda\perp})$$

where λ is a p-regular partition.

Theorem 2.10. *(James) As λ varies over p-regular partitions of n, D^λ varies over a complete set of inequivalent irreducible kS_n-modules. Each D^λ is self-dual and absolutely irreducible. Every field is a splitting field for S_n.*

For a proof of this result, see [7].

Example 2.11. Let $\lambda = (n)$ be the partition with just one row of length n. Then all λ-tableaux are row equivalent, so there is only one λ-tabloid, and $M^{(n)} = k$ is the one dimensional trivial module. We also have that $S^{(n)} = D^{(n)} = k$.

Of all the other M^λ modules, the easiest to deal with are the ones given by two-part partitions. The following well-known result provides a way of visualizing these modules.

Lemma 2.12. *The module $M^{(n-i,i)}$ is isomorphic to the kS_n-permutation module of all subsets of $\{1, 2, \ldots, n\}$ of size i.*

Proof. The isomorphism is given by sending an $(n-i, i)$-tabloid to the set of numbers in its second row. □

In the very special case when $\lambda = (n-1, 1)$ (and only for the field of two elements) we additionally have

Lemma 2.13. *Let k be the field with two elemenst. The module $M^{(n-1,1)}$ is isomorphic to the module consisting of all subsets of $\{1, \ldots, n\}$, where addition is given by the symmetric difference of sets, and where S_n acts by permuting the elements of each set. The isomorphism takes $S^{(n-1,1)}$ to the family of subsets of $\{1, \ldots, n\}$ of even cardinality. If n is odd, then $S^{(n-1,1)} = D^{(n-1,1)}$. If n is even, then $D^{(n-1,1)}$ is isomorphic to the quotient of the family of subsets of even cardinality modulo the equivalence relation that pairs a set with its complement.*

Proof. Notice that the family of all subsets of $\{1, \ldots, n\}$ is a vector space over the field of two elements, and that the singletons are a basis permuted by S_n. The isomorphism between $M^{(n-1,1)}$ and this module is given (as before) by sending the basis of $(n-1, 1)$-tabloids to their respective second rows. Note that the elements of $S^{(n-1,1)}$ are precisely the linear combinations of an even number of tabloids, which correspond to the subsets of even cardinality. Furthermore, the bilinear form on the subsets A and B now is given by the cardinality of $A \cap B$ modulo 2. The only non-empty subset that can be orthogonal to all subsets of even cardinality is $\{1, \ldots, n\}$, which is in $S^{(n-1,1)}$ if and only if n is an even number. Thus $D^{(n-1,1)} = S^{(n-1,1)}$ if n is odd, and is as described in the statement of the Lemma when n is even. □

In this paper we shall only work with the field of two elements.

We now record a number of well-known results which can be found either explicitly or implicitly in [7]. Our goal is to obtain decomposition formulas for $M^{(n-1,1)}$ and $M^{(n-2,2)}$

Corollary 2.14. *Let k be the field with two elemenst. If n is odd then*

$$M^{(n-1,1)} = D^{(n-1,1)} \oplus k$$

Proof. We already know that $D^{(n-1,1)} = S^{(n-1,1)}$ is a submodule of codimension 1. The complement is generated by the vector $\{1, \ldots, n\}$, where we have identified $M^{(n-1,1)}$ with the power set of $\{1, \ldots, n\}$. □

Now we proceed to derive a similar formula for $M^{(n-2,2)}$.

Lemma 2.15. *Let k be the field with two elemenst. The image of the map*

$$\varphi : M^{(n-2,2)} \longrightarrow M^{(n-1,1)}, \quad \{\alpha, \beta\} \mapsto \{\alpha\} + \{\beta\}$$

is $S^{(n-1,1)}$. The kernel of φ contains $S^{(n-2,2)}$ as a submodule of codimension one.

Proof. Now we switch back to the notation suggested by Lemma 2.12, not by Lemma 2.13. We can see that φ is a well-defined morphism of kS_n-modules. The image of φ is generated by all elements of the form $\{\alpha\} + \{\beta\}$ (with $\alpha \neq \beta$), which also generate $S^{(n-1,1)}$ (they are the polytabloids). There are module generators (polytabloids) of $S^{(n-2,2)}$ of the form $\{2, \beta\} + \{1, \beta\} + \{2, 3\} + \{1, 3\}$ or of the form $\{\alpha, \beta\} + \{1, \beta\} + \{\alpha, 2\} + \{1, 2\}$; both of them will go to zero under φ. Finally, notice that the kernel of φ has dimension $\frac{n(n-1)}{2} - (n-1) = \frac{n(n-3)}{2} + 1$, which is one more than the dimension of $S^{(n-2,2)}$ (see the Hook Formula, result 20.1 in [7] for the dimension of the Specht modules). □

Lemma 2.16. *Let k be the field with two elemenst. When $n \equiv 3 (\bmod 4)$, we have that $S^{(n-2,2)} = D^{(n-2,2)}$.*

Proof. This follows from Theorem 23.13 in [7]. □

Now we can quickly get the composition factors of $M^{(n-2,2)}$.

Corollary 2.17. *Let k be the field with two elemenst. When $n \equiv 3 (\bmod 4)$ we have that the composition factors of the module $M^{(n-2,2)}$ are k, $D^{(n-1,1)}$ and $D^{(n-2,2)}$.*

Proof. The image of φ is $S^{(n-1,1)} \cong D^{(n-1,1)}$ (because n is odd) and the kernel has k and $S^{(n-2,2)} \cong D^{(n-2,2)}$ (because n is congruent to 3 modulo 4) as composition factors. □

Our next result is a general statement about modules for arbitrary rings

Proposition 2.18. *Let R be a ring, M an R-module, S a simple R-module which is a composition factor of M of multiplicity 1. Then S is a direct summand of M if and only if there exist non-zero maps $\theta : S \longrightarrow M$ and $\phi : M \longrightarrow S$. Moreover, in the latter case, such maps must necessarily split.*

Proof. If S is a direct summand of M, then let θ and ϕ be the usual inclusion and projection respectively. Now assume the non-zero maps θ and ϕ exist. We must show that both θ and ϕ are split. Notice that $\phi\theta \neq 0$ (otherwise the series $0 < Im(\theta) \leq Ker(\phi) < M$ would give two composition factors isomorphic to S), so $\phi\theta$ must be an automorphism of S. This proves that ϕ is a split epimorphism and θ is a split monomorphism. □

We can combine the previous results to get

Proposition 2.19. *Let k be the field with two elemenst. When n is congruent to 3 modulo 4 we have*

$$M^{(n-2,2)} \cong D^{(n-2,2)} \oplus k \oplus D^{(n-1,1)}.$$

Proof. We know that $M^{(n-2,2)}$ has three different composition factors, so we can apply the previous proposition. The map φ gives rise to a morphism from $M^{(n-2,2)}$ onto $D^{(n-1,1)}$, and both modules are self-dual, so there is a non-zero map from $D^{(n-1,1)}$ to $M^{(n-2,2)}$. By the proposition φ splits, so $M^{(n-2,2)}$ is the direct sum of $D^{(n-1,1)}$ and the kernel of φ. Similarly, $D^{(n-2,2)}$ is a submodule of the kernel of φ, so $D^{(n-2,2)}$ is embedded in $M^{(n-2,2)}$ and both modules are self-dual, so there exists an epimorphism from $M^{(n-2,2)}$ onto $D^{(n-2,2)}$. By the proposition the inclusion of $D^{(n-2,2)}$ into $M^{(n-2,2)}$ is split, so in particular the inclusion of $D^{(n-2,2)}$ into the kernel of φ is split, with k as a direct complement. □

Now this decomposition theorem follows easily.

Corollary 2.20. *Let k be the field with two elemenst. Let n be congruent to 3 modulo 4. There is a decomposition*

$$M^{(n-2,2)} \cong D^{(n-2,2)} \oplus M^{(n-1,1)}.$$

Proof. It follows from the previous proposition and the fact that $M^{(n-1,1)} \cong D^{(n-1,1)} \oplus k$ because n is odd. □

Finally, we remind the reader of some useful facts about the representations of Cartesian products of groups.

Proposition 2.21. *Let k be a splitting field for the finite groups R and S, let U, T be finite dimensional modules for kR and kS respectively, and let $k(R \times S)$ act on $U \otimes_k T$ via $(r, s)(u \otimes t) = ru \otimes st$. Then*

(i) $U \otimes_k T$ is a simple $k[R \times S]$-module if and only if U is a simple kR-module and T is a simple kS-module.

(ii) $U \otimes_k T$ is a projective $k[R \times S]$-module if and only if U is a projective kR-module and T is a projective kS-module.

Proof. (i) A proof of this result can be found in [5], Theorem (10.33).

(ii) It is clear that the tensor product of two projective modules is projective. Assume that $U \otimes T$ is a projective module for the group $R \times S$. Then its restriction to the subgroup $R = R \times \{1\}$ is projective, and this is isomorphic to several copies of the R-module U (as many copies as the dimension of T over k), which proves that U must be projective. A similar argument proves that T is a projective kS-module. □

This situation arises naturally when a subgroup H of S_n has fixed points on the set $\{1, \ldots, n\}$.

Definition 2.22. Let X be a finite set. The set S_X is the group of permutations of X.

Lemma 2.23. *Let M be a finite set. Let H be a subgroup of the symmetric group S_M with fixed points F and let Θ be the complement of F in M. Then*

$$N_{S_M}(H)/H = (N_{S_\Theta}(H)/H) \times S_F$$

Proof. One containment is immediate. Now let $\tau \in N_{S_M}(H)$. Then τ permutes the fixed points F of H, so $\tau = \alpha\beta$ with $\alpha \in S_\Theta$ and $\beta \in S_F$. It follows that α is in $N_{S_\Theta}(H)$. □

As a result, a module for the quotient group $N_{S_M}(H)/H$ is really a module for the product $(N_{S_\Theta}(H)/H) \times S_F$. In this context, if we refer to a kS_F-module U as a $k[N_{S_M}(H)/H]$-module, we mean $k \otimes_k U$, that is, $N_{S_\Theta}(H)/H$ acts trivially on U. Notice that the $k[N_{S_M}(H)/H]$-module $k \otimes_k U$ will be simple and projective if and only if both k and U are simple and projective modules for $N_{S_\Theta}(H)/H$ and S_F respectively, that is, if and only if H is a Sylow 2-subgroup of S_Θ and U is a simple and projective kS_F-module. Note that the field F_2 is a splitting field for all symmetric groups, and the subgroups H we shall deal with are all weight subgroups of S_n, whose quotients $N_G(H)/H$ are isomorphic to cartesian products of general linear groups (see [3]), for which F_2 is also a splitting field.

3. Brauer quotient of $D^{(n-1,1)}$, n odd

We prove that the Brauer quotient of $D^{(n-1,1)}$ is either 0 or simple, and describe the subgroups H of S_n with respect to which the Brauer quotient is also projective as a module for $N_{S_n}(H)/H$. Note that the number of fixed points of any 2-subgroup H on the set $\{1,\dots,n\}$ must be odd because n is odd.

Theorem 3.1. *(*Webb, private communication*) Let k be the field of two elements, n an odd number greater than 1, H a 2-subgroup of S_n whose fixed points on the set $\{1,\dots,n\}$ are precisely $\{1,\dots,r\}$, $N = N_{S_n}(H)$ and V the simple module $D^{(n-1,1)}$. If $r = 1$ then $\overline{FP}_V(H) = 0$. If $r > 1$ then the natural quotient $N/H \longrightarrow S_r$ splits, and the action of N/H on $\overline{FP}_V(H)$ is isomorphic to that of S_r on $D^{(r-1,1)}$.*

Proof. We have that $M^{(n-1,1)} = D^{(n-1,1)} \oplus k$, so

$$\overline{FP}_{M^{(n-1,1)}}(H) \cong \overline{FP}_V(H) \oplus \overline{FP}_k(H)$$

But $M^{(n-1,1)}$ is an H-permutation module with permutation basis the set $\{1,\dots,n\}$ (as in Lemma 2.12), so its Brauer quotient with respect to H is the permutation module on the set $\{1,\dots,r\}$ (where N/H acts as S_r) by Proposition 2.7. If $r = 1$ then cancellation gives $\overline{FP}_V(H) = 0$. If $r > 1$ then the previous permutation module is isomorphic to $M^{(r-1,1)}$. This gives the formula

$$M^{(r-1,1)} \cong \overline{FP}_V(H) \oplus k$$

Since n is odd and H is a 2-subgroup of S_n it follows that r is odd, so by Corollary 2.14 we get $M^{(r-1,1)} = D^{(r-1,1)} \oplus k \cong \overline{FP}_V(H) \oplus k$ and cancellation gives $\overline{FP}_V(H) \cong D^{(r-1,1)}$. □

The following well-known result can be proved for all n using the Nakayama conjecture, or by looking at powers of 2 dividing the dimension of $D^{(n-1,1)}$ and $n!$, but it is also a consequence of our previous theorem when n is odd.

Corollary 3.2. *Let k be the field of two elements and let n be odd. The module $D^{(n-1,1)}$ is a projective kS_n-module if and only if $n = 3$.*

Proof. We know that $D^{(2,1)}$ is the two dimensional simple projective module of kS_3 (see [1]). If $n \geq 5$, then the 2-subgroup H generated by the transposition (1,2) has at least 3 fixed points, so by the theorem we have $\overline{FP}_{D^{(n-1,1)}}(H) \neq 0$, so $D^{(n-1,1)}$ cannot be projective (see Example 2.8). □

Since any 2-subgroup of S_n is conjugate to an H as described in Theorem 3.1, the condition that the fixed points be exactly $\{1, \ldots, r\}$ is not necessary.

Corollary 3.3. *Let k be the field with two elemenst. Let n be an odd number greater than 1, H a 2-subgroup of S_n. Then the Brauer quotient $\overline{FP}_{D^{(n-1,1)}}(H)$ is either 0 or a simple N/H-module. It is projective if and only if H is conjugate to a Sylow 2-subgroup of S_{n-3}. In this case, it is isomorphic to $D^{(2,1)}$*

Proof. Without loss of generality we may assume that the fixed points of H on the set $\{1, \ldots, n\}$ are precisely $\{1, \ldots, r\}$. The Brauer quotient is simple because $D^{(r-1,1)}$ is a simple S_r-module, and it is projective if and only if the kernel of the map $N/H \longrightarrow S_r$ has no elements of order 2 and $D^{(r-1,1)}$ is a projective S_r-module (since $N/H \cong \big(N_{S_{n-r}}(H)/H\big) \times S_r$). We know that $N_{S_{n-r}}(H)/H$ has even order if and only if H is not a Sylow 2-subgroup of S_{n-r} by a theorem about normalizers in p-groups, and that $D^{(r-1,1)}$ is projective precisely when $r = 3$. □

4. Brauer quotient of $D^{(n-2,2)}$, $n \equiv 3 \pmod 4$

We describe all the subgroups H of S_n such that the Brauer quotient of $D^{(n-2,2)}$ is both simple and projective as a module for $N_{S_n}(H)/H$.

In this section n is congruent to 3 modulo 4 and $n \geq 7$, k is the field of two elements, V is the simple module $D^{(n-2,2)}$, H is a 2-subgroup of S_n with exactly r fixed points on $\{1, \ldots, n\}$, $N = N_{S_n}(H)$, A is the family of H-orbits on $\{1, \ldots, n\}$ of size 2, B and C are the families of subsets of size 2 and 1 respectively of the fixed points of H on $\{1, \ldots, n\}$. As usual, if X is a set where a group acts, kX denotes the permutation module on X. Recall that $N/H \cong \big(N_{S_{n-r}}(H)/H\big) \times S_r$.

Proposition 4.1. *With the notation and hypotheses stated at the beginning of this section, we have that*
$$\overline{FP}_V(H) \oplus kC \cong kA \oplus kB$$
as N/H-modules.

Proof. We compute $\overline{FP}_{M^{(n-2,2)}}(H)$ in two different ways. From
$$M^{(n-2,2)} = D^{(n-2,2)} \oplus M^{(n-1,1)}$$
we get
$$\overline{FP}_{M^{(n-2,2)}}(H) \cong \overline{FP}_V(H) \oplus \overline{FP}_{M^{(n-1,1)}}(H) \cong \overline{FP}_V(H) \oplus kC$$
On the other hand, using the fact that $M^{(n-2,2)}$ is an H-permutation module we get that $\overline{FP}_{M^{(n-2,2)}}(H) \cong kA \oplus kB$, since the fixed points of H acting on the subsets of $\{1, \ldots, n\}$ of size 2 are made up of subsets fixed pointwise by H, and subsets forming a single H-orbit of size 2. □

This is not an explicit description of $\overline{FP}_V(H)$, but it is enough to determine when the Brauer quotient is a simple projective module.

Corollary 4.2. *If $r \neq 1$ then $\overline{FP}_V(H)$ is not a simple and projective $k[N/H]$-module. If $r = 1$, then $\overline{FP}_V(H)$ is a direct summand of kA of codimension 1 and the number of orbits of size 2 is odd.*

Proof. If $r \geq 5$ then $kB \cong M^{(r-2,2)}$, $kC \cong M^{(r-1,1)}$, and by Lemma 2.15, if $\overline{FP}_V(H)$ were simple and projective (counting composition factors) we would have $\overline{FP}_V(H) \cong D^{(n-2,2)}$ as N/H-modules, and $D^{(r-2,2)}$ (or rather, $k \otimes_k D^{(n-2,2)}$) is not a simple projective $k[N/H]$-module. If $r = 3$ then $kB \cong kC$ (subsets of size 2 and 1 will give isomorphic families), so $\overline{FP}_V(H)$ is isomorphic to the permutation module kA, which can be simple only if it is one-dimensional, which implies that H has exactly 3 fixed points and one orbit of size 2 on $\{1, \dots, n\}$. So n is congruent to 1 modulo 4, contradicting our hypothesis. If $r = 1$, then $kC = k$, $kB = 0$, and the number of orbits of size 2 must be odd because $n \equiv 3(\bmod 4)$. □

Remark 4.3. For the remainder of this section we shall assume that $r = 1$ and that H has at least 3 orbits of size 2 (if it had only 1 then kA would have dimension 1 and $\overline{FP}_V(H) = 0$).

We must determine what the action of N/H on kA is, that is, how N/H acts on the orbits of size 2 of H on $\{1, \dots, n\}$. Without loss of generality, let $\{i, l+i\}_{i=1}^{l}$ be the orbits of H of size 2 (where l is the number of such orbits).

Lemma 4.4. *If kA has a projective non-trivial summand (as a module for N/H) then H contains the transpositions $(i, l+i)$, $1 \leq i \leq l$ and a Sylow 2-subgroup of $S_{\{2l+1,\dots,n\}}$.*

Proof. If $(i, l+i) \notin H$, then it represents an element of order 2 in N/H that acts trivially on the set A, hence on the module kA, hence on its direct summands, which cannot be projective. A similar argument proves that H must contain a Sylow 2-subgroup of $S_{\{2l+1,\dots,n\}}$. □

Corollary 4.5. *If kA has a projective non-trivial summand (when viewed as a module for N/H), then H is the internal direct product of the elementary abelian subgroup generated by the transpositions $(i, l+i)$ $1 \leq i \leq l$ and a Sylow 2-subgroup of $S_{\{2l+1,\dots,n\}}$.*

Proof. The lemma proves one containment. The other follows from the fact that H is a 2-subgroup and it is therefore contained in the direct product of some Sylow 2-subgroups of the symmetric groups on its orbits, which in this case are generated by the elements mentioned. □

Proposition 4.6. *Let m be a natural number, E the subgroup of S_m generated by the disjoint transpositions $\{(a_i, b_i) \mid 1 \leq i \leq l\}$, and X the complement of the previous $2l$ points. Let Λ be a subgroup of S_X, put $Q = E \cdot \Lambda$ and let $N = N_{S_m}(Q)$. Then the action of N on the set of l pairs $\{a_i, b_i\}$ induces a surjective morphism $\phi : N \longrightarrow S_l$ whose kernel is $E \cdot N_{S_X}(\Lambda)$.*

Proof. The morphism ϕ induced by the action of N on the l pairs is well defined because N sends Q-orbits to Q-orbits. In order to prove that ϕ is surjective, it suffices to cover all transpositions in S_l. If $\{a, b\}$ and $\{c, d\}$ are two of the l pairs, then $(a, c)(b, d)$ is an element in N whose action on the pairs is that of the transposition we wanted. Now let us determine the kernel of ϕ. We have that that $E \cdot N_{S_X}(\Lambda)$ is in the kernel of ϕ. Now let σ be in the kernel of ϕ. Then σ preserves the l pairs, so $\sigma = \varepsilon\delta$ with $\varepsilon \in E$ and $\delta \in S_X$, so $\delta = \varepsilon^{-1}\sigma \in N \cap S_X \subset N_{S_X}(\Lambda)$. □

Corollary 4.7. *We have that*

$$N/H \cong S_l$$

where l is the number of H-orbits of size 2.

Proof. By Proposition 4.6, $\phi : N \longrightarrow S_l$ is surjective, and since $N_{S_X}(P) = P$ where P is a Sylow 2-subgroup of $S_{\{2l+1,\dots,n\}}$ contained in H, then the kernel of ϕ is H. □

Corollary 4.8. *Let l be the number of H-orbits of size 2. Then $kA \cong M^{(l-1,1)}$ and $\overline{FP}_V(H) \cong D^{(l-1,1)}$ as S_l-modules.*

Proof. The morphism $\phi : N \longrightarrow S_l$ that induces the isomorphism between N/H and S_l takes the permutation module of H-orbits of size 2 (i.e. kA) to the permutation module arising from the set $\{1,\dots,l\}$, (i.e., $M^{(l-1,1)}$). We also know that l is odd, so $M^{(l-1,1)} \cong D^{(l-1,1)} \oplus k$. Now apply Corollary 4.2. □

Theorem 4.9. *Let k be the field with two elements. Let n be congruent to 3 modulo 4, and greater than or equal to 7. Let V be the simple kS_n-module $D^{(n-2,2)}$, and H a 2-subgroup of S_n with normalizer N. We have that $\overline{FP}_V(H)$ is simple and projective if and only if H is conjugate to a subgroup of the form $E \cdot P$ where E is generated by $(1,2),(3,4),(5,6)$, and P is a Sylow 2-subgroup of $S_{\{7,\dots,n\}}$. In this case, $N/H \cong S_3$ and $\overline{FP}_V(H) \cong D^{(2,1)}$.*

Proof. By Corollary 4.8 $\overline{FP}_V(H) \cong D^{(l-1,1)}$ as S_l-modules. The module $D^{(l-1,1)}$ is S_l-projective if and only if $l = 3$. This and Corollary 4.5 determine the structure of H. In this case, from Corollary 4.7 it follows that $N/H \cong S_3$, and by Corollary 4.8 we have that $\overline{FP}_V(H) \cong D^{(2,1)}$ is simple and projective. □

5. Brauer quotients of $S^{(n-1,1)}$ and $D^{(n-1,1)}$, $n \equiv 2 \pmod 4$

We describe the subgroups H of S_n (when n is congruent to 2 modulo 4) with respect to which the Brauer quotient of $S^{(n-1,1)}$ is simple and projective as a module for $N_{S_n}(H)/H$, and similarly for $D^{(n-1,1)}$. The module $S^{(n-1,1)}$ is not simple, but we need it to get to $D^{(n-1,1)}$, and the determination of its Brauer quotient is an easy by-product. In this section, k will be the field with two elements, n a number congruent to 2 modulo 4, H a 2-subgroup of S_n, N the normalizer of H in S_n, K a maximal subgroup of H, V the Specht module $S^{(n-1,1)}$ and W the irreducible module $D^{(n-1,1)}$. From now on we shall use the characterization of V and W as families of subsets under symmetric differences (see Lemma 2.13). Note that $V/X \cong W$ where X be the vector subspace of V spanned by the vector $\{1,\dots,n\}$.

In order to obtain the Brauer quotient, first we need to describe the fixed points V^H and W^H.

Proposition 5.1. *With the notation and hypotheses stated at the beginning of this section, we have that the fixed points V^H of H on V are the H-invariant subsets of $\{1,\dots,n\}$ of even cardinality. The fixed points W^H of H on W are the classes of elements in V^H modulo X.*

Proof. The statement for V^H follows from the definition of fixed points and the characterization of V. Now let A be a subset representing an element in W that is fixed by all the elements in H, and let $h \in H$. Then $h(A)$ is either A or its complement. Suppose $h(A) = A^c$. Then both subsets have the same cardinality, namely $n/2$, which is an odd number (because n is congruent to 2 modulo 4), and this contradicts the fact that A has even cardinality (since it is in V). Thus $h(A) = A$, which proves the assertion about the fixed points of H on W. □

Large orbits (of size 4 or larger) are in the image of relative traces from proper subgroups, so they will not contribute towards the Brauer quotient.

Proposition 5.2. *Let $\Omega \in V^H$ be an H-orbit whose size is divisible by 4. Then there is a maximal subgroup K of H and $\Lambda \in V^K$ such that $tr_K^H(\Lambda) = \Omega$. If $[\Omega]$ and $[\Lambda]$ are the corresponding classes in W^H and W^K, then $tr_K^H([\Lambda]) = [\Omega]$.*

Proof. We have that Ω is isomorphic to an H-set of the form H/L where L is a subgroup whose index in H is at least 4. Let K be any maximal subgroup of H containing L. Then Ω is a union of two K-orbits of equal size, say Λ and $h\Lambda$, where $h \in H$ but $h \notin K$. Since Ω has size divisible by 4, we have that Λ has even size, so $\Lambda \in V^K$. Furthermore, $tr_K^H(\Lambda) = \Lambda + h\Lambda = \Lambda \triangle h\Lambda = \Lambda \coprod h\Lambda = \Omega$. We also have $tr_K^H([\Lambda]) = [tr_K^H(\Lambda)] = [\Omega]$. □

Remark 5.3. Since all H-orbits of size divisible by 4 are in the image of a relative trace map from a proper subgroup, the Brauer quotient (of V and W) with respect to H is a quotient of the space spanned by (classes of) fixed points and orbits of size 2 of H. Now we proceed to determine which of these orbits will also be in the image of a relative trace from a proper subgroup. Since $tr_K^H \circ tr_L^K = tr_L^H$, it suffices to consider maximal subgroups of H.

Lemma 5.4. *Let A be the set of K-fixed points that are not fixed by H, and let h be an element in H that is not in K. Then h acts on A as a product of disjoint transpositions. These transpositions are independent of the choice of h, and the pairs they determine are H-orbits of size 2.*

Proof. We have that K is normal in H, so h permutes the K fixed points without fixing any of them (otherwise that point would also be fixed by H), but $h^2 \in K$, so h^2 acts trivially on A, and this implies that h acts on A as a product of disjoint transpositions. The rest follows from the fact that K has index 2 in H, so any other representative is of the form hk with $k \in K$. □

We study in greater detail these special orbits of size two.

Definition 5.5. Let B and C be H-orbits. We say they are *glued* if for every h in H, h fixes B pointwise if and only if h fixes C pointwise.

Lemma 5.6. *Gluing is an equivalence relation in the set of all H-orbits of size 2.*

Proof. This is immediate from the definition. □

We also know that the normalizer of H sends glued orbits to glued orbits.

Lemma 5.7. *Let D, E be glued orbits of H, and let $n \in N$. Then nD, nE are also glued H-orbits. In particular, N permutes the equivalence classes of orbits of H under gluing, and it preserves the cardinality of each equivalence class.*

Proof. Let $h \in H$, Then h fixes D pointwise if and only if $hd = d$ for all $d \in D$, if and only if $nhn^{-1}(nd) = nd$ for all $d \in D$, if and only if nhn^{-1} fixes nD pointwise. In other words, nhn^{-1} fixes nD pointwise if and only if h fixes D pointwise, if and only if h fixes E pointwise, if and only if nhn^{-1} fixes nE pointwise, so nD and nE are glued nHn^{-1}-orbits. Since $n \in N$, we have the desired result. □

Remark 5.8. The H-orbits of size 2 described in Lemma 5.4 are glued. Moreover, they form a single equivalence class. The following result proves that this is always how these equivalence classes arise.

Lemma 5.9. *Let A be the union of an equivalence class of H-orbits of size 2 under gluing. Let L be the pointwise stabilizer of A in H. Then L is maximal in H.*

Proof. Let x be any element in H that is not in L. Then x must act on A as a product of disjoint transpositions without fixed points. If y is another element in H that is not in L, then y also acts as the same product of transpositions, so xy acts as the identity on those points, and it is in L. This proves that L has index 2 in H. □

Corollary 5.10. *The maximal subgroups of H which have strictly more fixed points than H are precisely the ones described in Lemma 5.9.*

Proof. This follows from Lemmas 5.4 and 5.9. □

Next we prove that the Brauer quotients of V and W cannot be simple and projective unless H acts fixed-point freely on the set $\{1, \ldots, n\}$.

Theorem 5.11. *Assume that H has at least one fixed point on the set $\{1, \ldots, n\}$, say a. Then the images of the relative traces from proper subgroups of H are the subspace of V^H generated by all the orbits of H on $\{1, \ldots, n\}$ of even cardinality. The images of the relative traces in W^H are the classes of such orbits of even cardinality modulo X.*

Proof. By Proposition 5.2, every orbit of H on $\{1, \ldots, n\}$ whose size is divisible by 4 will be in the image of a relative trace from a proper subgroup. Let $\{\alpha, \beta\}$ be an H-orbit of size 2. Since we are assuming that a is a fixed point of H, we must have $\alpha \neq a \neq \beta$. Let L be the maximal subgroup of H fixing α and β (see Lemma 5.9), and let $h \in H$, $h \notin L$, so by Lemma 5.4 $h\alpha = \beta$. We have that $\{a, \alpha\} \in V^L$, and $tr_L^H(\{a, \alpha\}) = \{a, \alpha\} + \{a, \beta\} = \{\alpha, \beta\}$. In W, we have that $[\{a, \alpha\}] \in W^L$ and $tr_L^H([\{a, \alpha\}]) = [\{\alpha, \beta\}]$. The other containment follows from the fact that if L is maximal in H, $\Omega \in V^H$ and x is a point fixed by H with $x \in \Omega$, then $x \in h\Omega$ for every $h \in H$, so $x \notin tr_L^H(\Omega)$, and the image of a relative trace from a maximal subgroup cannot contain any points fixed by H. If $[\Omega] \in W^L$ with $\Omega \in V^L$ then $tr_L^H([\Omega]) = [tr_L^H(\Omega)]$, so $tr_L^H(W^L)$ consists of the classes of the elements in $tr_L^H(V^L)$. □

Corollary 5.12. *Assume that the fixed points of H on $\{1, \ldots, n\}$ are precisely $\{1, \ldots, r\}$. Then $\overline{FP}_V(H) \cong S^{(r-1,1)}$ and $\overline{FP}_W(H) \cong D^{(r-1,1)}$, where N/H acts via the canonical quotient map $N/H \longrightarrow S_r$. In particular, neither $\overline{FP}_V(H)$ nor $\overline{FP}_W(H)$ is a simple projective module.*

Proof. Note that r is even because n is even and H is a 2-group. The Brauer quotient $\overline{FP}_V(H)$ is the family of H-invariant subsets of $\{1, \ldots, n\}$ of even cardinality modulo the span of the orbits of H of even cardinality. Let $\varphi : V^H \longrightarrow M^{(r-1,1)}$ be the morphism of kS_r-modules sending a set Ω to the set of fixed points of H in Ω (we are again using the description of $M^{(r-1,1)}$ given in terms of subsets of $\{1, \ldots, n\}$ from Lemma 2.13). Then the kernel of φ consists of the H-invariant subsets of even cardinality with no fixed points, which is the same as the vector subspace spanned by the H-orbits of even cardinality, so $\overline{FP}_V(H) \cong \varphi(V^H)$, and since an H-invariant subset of even cardinality must have an even number of fixed points, we have $\varphi(V^H) = S^{(r-1,1)}$. This in turn induces an isomorphism between $\overline{FP}_W(H) \cong \overline{FP}_V(H)/ < [\{1,\ldots,n\}] >$ and $S^{(r-1,1)}/ < \varphi(\{1,\ldots,n\}) >= S^{(r-1,1)}/ < \{1,\ldots,r\} >= D^{(r-1,1)}$. Since r is even, $(r-1,1)$ cannot be a triangular partition, so neither $S^{(r-1,1)}$ nor $D^{(r-1,1)}$ is a simple and projective module for kS_r. $\square$

This means that if we want a simple projective Brauer quotient, we must only consider subgroups H with no fixed points on $\{1,\ldots,n\}$.

Theorem 5.13. *Assume H has no fixed points. Then the images of the relative traces generate the same module as the (classes of) orbits of size divisible by 4 together with the sums of pairs of glued H-orbits of size 2 (i.e. subsets $\{a,b,c,d\}$ where $\{a,b\}$, $\{c,d\}$ are glued H-orbits of size 2).*

Proof. By Proposition 5.2, any orbit whose size is divisible by 4 will be in the image of a relative trace. Each element in V^K is a K-invariant subset of $\{1,\ldots,n\}$, and can be written as a disjoint union of K-orbits. If Ω is a K-orbit with $|\Omega| \geq 2$ and $h \in H$, $h \notin K$, then $tr_K^H(\Omega) = \Omega + h\Omega = \Omega \triangle h\Omega$. Let $w \in \Omega$, so $\Omega = Kw$, and $h\Omega = hKw = hKh^{-1}hw = K(hw)$ is another K-orbit of the same size, so either $h\Omega = \Omega$ (and $\Omega\triangle h\Omega = \emptyset$) or $\Omega \cap h\Omega = \emptyset$ (and $\Omega\triangle h\Omega$ is an H-orbit of size $2|\Omega|$). We see that the only possible orbits of size 2 in the images of the relative traces are the images of these maps on subsets of the K-fixed points of maximal subgroups K. The only possible fixed points for K will not be fixed by H, so they come from glued H-orbits of size 2 (see Remark 5.8). Recall that the subsets of even cardinality are generated by all possible pairs. A pair of fixed points under K is either an H-orbit, or they lie in distinct H-orbits which are glued. Let $\{a,b\}$ and $\{c,d\}$ be glued H-orbits, and let K be their corresponding maximal subgroup. If they are different orbits, then $tr_K^H(\{a,c\}) = \{a,c,b,d\}$, but using the same orbit gives $tr_K^H(\{a,b\}) = 0$, so the 2-orbits in the image of this relative trace are the sums of pairs of different glued H-orbits of size 2. $\square$

Now we know that the Brauer quotients depend on the glued orbits of size 2. We use this to give another description of the Brauer quotients in terms of certain subspaces of the fixed points.

Lemma 5.14. *Assume that H has no fixed points on $\{1,\ldots,n\}$. Recall that X denotes the subspace of V generated by the vector $\{1,\ldots,n\}$, so X is also a subspace of V^H. Let Y be the subspace of V^H spanned by the H-orbits of size 2, and let Z be the subspace of Y spanned by all sums of pairs of glued H-orbits of size 2. Let Λ be the vector subspace of V^H spanned by all the H-orbits of size greater than or equal to 4, and let w be the sum of all the orbits of size 2. We have that:*

(i) The dimension of Y equals the number of H-orbits of size 2.

(ii) The codimension of Z in Y equals the number of equivalence classes of orbits of size 2 under gluing.

(iii) $V^H = Y \oplus \Lambda$

(iv) $W^H = V^H/X = (Y \oplus \Lambda)/X$

(v) $\sum tr_K^H(V^K) = Z \oplus \Lambda$

(vi) $\sum tr_K^H(W^K) = (Z + \Lambda + X)/X$

(vii) $\overline{FP}_V(H) \cong Y/Z$ as $k[N/H]$-modules.

(viii) $\overline{FP}_W(H) \cong Y/(Z+ < w >)$ as $k[N/H]$-modules.

Proof. (i) is immediate, and (ii) follows from the fact that if $v_1, \ldots, v_t$ is a basis of a vector space, the subspace generated by all the sums $v_i + v_j$ with $i \neq j$ has codimension 1. (iii) follows from the fact that an H-invariant subset of $\{1, \ldots, n\}$ can be written uniquely as a disjoint union of orbits, and (iv) is the description of W^H from Proposition 5.1. (v) and (vi) are given by Theorem 5.13 (notice that we do not know whether $\{1, \ldots, n\} \in Z \oplus \Lambda$ or not, so the sum $Z + \Lambda + X$ may not be direct). We have $\overline{FP}_V(H) = V^H / \sum tr_K^H(V^K) \cong (Y \oplus \Lambda)/(Z \oplus \Lambda) \cong Y/Z$, which proves (vii). We use one of the isomorphism theorems to get $\overline{FP}_W(H) = W^H / \sum_K^H(W^K) \cong [(Y \oplus \Lambda)/X] / [(Z + \Lambda + X)/X] \cong (Y \oplus \Lambda)/(Z + \Lambda + X)$. Finally, consider the composition of morphisms of $k[N/H]$-modules $Y \longrightarrow Y \oplus \Lambda \longrightarrow (Y \oplus \Lambda)/(Z + \Lambda + X)$. Note that this is surjective, and that its kernel is $Y \cap (Z + \Lambda + X) = Z + < w >$, so we have (viii). □

We proceed to study the modules V and W separately.

Theorem 5.15. *Assume H has no fixed points. Let $A_1, \ldots, A_r$ be the equivalence classes of 2-orbits under gluing. Then N permutes the A_i, and the Brauer quotient of V with respect to H is isomorphic to the permutation module on $A_1, \ldots, A_r$.*

Proof. We use the notation from Lemma 5.14. Let φ be the map from the permutation module of the A_i into the Brauer quotient given by sending A_i to the class of any of its representatives. This map is well defined, because the difference of two representatives is in Z. It is also surjective, because any orbit of size 2 lies in one of the A_i. By Lemma 5.14 (ii) and (vii), the dimension of $\overline{FP}_V(H)$ is r, so the domain and image of φ have the same dimension, so φ is an isomorphism. □

Theorem 5.16. *Let k be the field with two elements, n a number congruent to 2 modulo 4, H a 2-subgroup of S_n, N the normalizer of H in S_n and V the Specht module $S^{(n-1,1)}$. We have that $\overline{FP}_V(H)$ is simple and projective if and only if H is a Sylow 2-subgroup of S_n. In this case, $N = H$, and $\overline{FP}_V(H)$ is the 1-dimensional trivial module.*

Proof. If H is a Sylow 2-subgroup then it has no fixed points and it has a unique orbit of size 2 (because $n \equiv 2 (\bmod 4)$), and we have the desired result. Now suppose H is a 2-subgroup with simple projective Brauer quotient. By Corollary 5.12 H cannot have any fixed points, and by Theorem 5.15 the Brauer quotient is a permutation module, which is simple only when 1 dimensional trivial, and this is in turn projective only for a group of odd order, so H must be a Sylow 2-subgroup of S_n. □

Now for $W = D^{(n-1,1)}$.

Theorem 5.17. *Assume H has no fixed points. Let $A_1, \ldots, A_r$ be the equivalence classes of 2-orbits under gluing. Then N permutes the A_i, and the Brauer quotient $\overline{FP}_W(H)$ is isomorphic to the quotient of the permutation module on $A_1, \ldots, A_r$ by the submodule spanned by the element $\sum_{i=1}^r |A_i| A_i$.*

Proof. Once again we use the notation from Lemma 5.14. Let φ be the map from the permutation module of the A_i into the module $Y/(Z + < w >)$ given by sending A_i to the class of any of its representatives. This map is well defined, because the difference of two representatives is in Z. It is also surjective, because any orbit of size 2 lies in one of the A_i. We can also see that the element $f := \sum_{i=1}^r |A_i| A_i$ is in the kernel of φ because $\varphi(f) = \sum_{i=1}^{r} |A_i|\varphi(A_i) = w$. Note that a sum of 2-orbits is in Z if and only if it has an even number of elements in each A_i, so $w \in Z$ if and only if $f = 0$. By Lemma 5.14 (ii) and (viii), the dimension of $\overline{FP}_W(H)$ is either r if $w \in Z$, or $r-1$ if $w \notin Z$. There are two cases:

Case 1: $w \in Z$. Then $f = 0$, the dimension of the Brauer quotient is r and φ is an isomorphism, so its kernel is spanned by f and the result holds.

Case 2: $w \notin Z$. Then $f \neq 0$, the dimension of the Brauer quotient is $r-1$ and the kernel of φ has dimension 1, so it is spanned by f, and the result holds. □

Theorem 5.18. *Let k be the field with two elements, n a number congruent to 2 modulo 4, H a 2-subgroup of S_n, N the normalizer of H in S_n and W the irreducible module $D^{(n-1,1)}$. We have that $\overline{FP}_W(H)$ is simple and projective if and only if H is conjugate to a subgroup of the form $< (1,2),(3,4),(5,6) > \times P$, where P is a Sylow 2-subgroup of $S_{\{7,\ldots,n\}}$. In this case, N/H is isomorphic to S_3, and $\overline{FP}_W(H)$ is isomorphic to $D^{(2,1)}$.*

Proof. It is immediate from Theorem 5.17 that the Brauer quotient of W with respect to the subgroup $H =< (1,2),(3,4),(5,6) > \times P$ is isomorphic to $M^{(2,1)}/k \cong D^{(2,1)}$ and that $N/H \cong S_3$. Suppose, conversely, that $\overline{FP}_W(H)$ is simple and projective. By Corollary 5.12, H cannot have any fixed points, so the conditions for Theorem 5.17 are satisfied. We also claim that no two different H-orbits of size 2 can be glued (if $\{a,b\}$ is glued to $\{c,d\}$, then $(a,c)(b,d)$ is in N but not in H, has order 2, and acts trivially on $\overline{FP}_W(H)$, which cannot be N/H-projective). The subgroup H must be contained in a subgroup of the form $E \cdot P$ where E is the subgroup generated by the transpositions (a_i, b_i) for all 2-orbits of H, and P is a Sylow 2-subgroup of the symmetric group on the remaining points. Note that $E \cdot P$ is a 2-subgroup containing H that acts trivially on the Brauer quotient. If H were properly contained in $E \cdot P$, then there would be an element in N/H of order 2 acting trivially on a projective module, so $H = E \cdot P$. Let $A_1, \ldots, A_r$ be the H-orbits of size 2. The action of N on the A_i induces a surjective morphism of groups to S_r whose kernel is H (see Proposition 4.6). Then the quotient N/H is isomorphic to S_r where r is the number of H-orbits of size 2, and the isomorphism is given by the action of N on such orbits. Furthermore, $\overline{FP}_W(H)$ is, as an N/H-module, isomorphic to the quotient of the permutation module $k\{1,\ldots,r\}$ modulo the sum of the basic elements, and this module is $D^{(r-1,1)}$. This is kS_r-projective if and only if $r = 3$, so H and $\overline{FP}_W(H)$ have the desired form. □

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