CS3001, Algorithm Design and Analysis
Tutorial 1

1. a) Prove the following theorem of Nicomachus (A.D. 100): $1^3 = 1$, $2^3 = 3 + 5$, $3^3 = 7 + 9 + 11$, $4^3 = 13 + 15 + 17 + 19$ etc.
b) Use this result to prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

2. Show that in every graph $G = (V, E)$ the number of vertices $v \in V$ with $\deg(V)$ odd is even.

3. What is the error in the following induction "proof"?
Theorem: Let S be a set of n numbers. Then all numbers in S are the same.
Proof: Proceed by induction over n:

> **Induction Hypothesis:**The theorem is true for all sets with n elements.

The hypothesis certainly holds for $n = 1$.
Let $n > 1$, we show that the hypothesis holds for $n + 1$: Let S be a set of size $n + 1$ and let $x, y \in S$ be two arbitrary elements. We have to show that $x = y$. Take $T = S \setminus \{x, y\}$ and let $T_x := T \cup \{x\}$, $T_y = T \cup \{y\}$. Then $|T_x| = |T_y| = n$, so by the hypothesis, x is equal to all elements in T. Similarly y is equal to all elements in T. Therefore $x = y$.

4. Prove by induction that for $n \geq 10$ we have $2^n > n^3$.

5. (Long integer multiplication) Let a, b two integers with n digits each. We assume that we can only multiply single digits. Let $T(n)$ be the runtime for the pencil-and-paper algorithm for multiplying numbers. Give an O-estimate for $T(n)$ (i.e. find $f(n)$ such that $T(n) = O(f(n))$).