

CS3001, Algorithm Design and Analysis

Comments and Sample solution

In various cases the argumentation in the solutions handed in was weak, while the basic idea was right (as far as I could understand the argumentation): A solution to an exercise consists of more than a line (or two lines) that give the result. It should also explain how the person arrived at this solution. This does not need to be a full description of the technique, references (like: "using index shifting we get ...") are quite sufficient but there should be sufficient information to permit the reader to follow your work. A sequence of algebraic expressions usually does not fulfill this condition.

If the solution contains text *read it through* at the end to make sure that the text is understandable (can be parsed). Punctuation between sentences can be of substantial help. If you use variables or expressions (like a and b in $an + b$) that are not given in the exercise say what they stand for.

6) Almost all solutions gave the right ordering. However the reasoning was often very short or missing. If you claim that $f(n) < g(n)$ for large n either show it, or give a reference to the theorem in the lecture you used.

Calculations in b) and c) were essentially correct (I have accepted any base of logarithm). Most solutions for b) however gave no reasoning why only the given range for n was considered.

7) If an inner loop runs to the index given by an outer loop the runtime is $\sum_{i=1}^n T(i)$ and not $nT(n)$. Asymptotically it will be the same, but to prove this in special cases was the aim of the exercise.

The O -notation is only a shorthand for eventual inequality. There might be occasions when lemmas are stated sloppily like $\sum O(x^i) = O(x^n + 1)$, but this does not mean that you can do algebra with O in the normal way.

To shorten notation statements or algorithms are often expressed in the form $P(n)$. Using the same expression for the runtime can lead to confusion.

8) While most solutions had the right idea, the argumentation was not always complete. (In particular the existence of "leaves" was not proven). Many solutions did not remove a vertex but built the graph. This approach is valid as well. However then you have to show that you can get **every** graph you want to consider (i.e. every tree or every graph with $|E| = |V| - 1$) in this way! (If we remove a vertex we know what we removed, we therefore can reconstruct the larger graph from the smaller one!)

Example solution:

a) We define a "leaf" to be a vertex of degree one.

Induction hypothesis: The theorem holds for connected graphs with n vertices.

The base case is trivial: There is only one graph with 1 vertex, it is a tree and has no edges, so for it $|E| = |V| - 1$ holds.

Now we prove the induction step. Assume that n is an integer such that the theorem holds for all graphs with less than n vertices. We have to show that the theorem holds for graphs with n vertices.

We have to show two directions:

i) " \Rightarrow :" Let $G = (V, E)$ be a tree with $n = |V|$ vertices. We show first that G has at least one leaf: We start with one vertex $v \in V$ and continue to go along edges without going over the same edge twice. As there is only a finite number of edges this process must stop at some point. The vertex w on which we then stop cannot have been visited on our way before – otherwise we would have discovered a cycle. As we had to stop on w , the degree of w is one. The subgraph $H = (W, F)$ induced by the subset $W = V \setminus \{w\}$ of vertices therefore is still a connected graph and $|W| = |V| - 1$, $|F| = |E| - 1$. It also does not have any cycles and thus is a tree with less than n vertices. The induction hypothesis shows that $|F| = |W| - 1$. We therefore have that $|E| = |F| + 1 = |W| - 1 + 1 = |V| - 1$.

ii) " \Leftarrow :" Now let $G = (V, E)$ be a connected graph with $n = |V| = |E| + 1$. If the degree of every vertex was at least 2, we would have that

$$2|E| = \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 2 = 2|V| = 2(|E| + 1),$$

contradiction. So there must be a leaf w in G . Again we form the (connected, as the edge removed connects only to a degree 1 vertex) subgraph $H = (W, F)$ induced by the subset $W = V \setminus \{w\}$ of vertices. It has less than n vertices, so by the induction hypothesis we know that H is a tree and thus has no cycles. But the only additional edge in G connects the degree-1 vertex w to the graph H . It therefore cannot be part of a cycle. Therefore every cycle in G would already be a cycle in H which shows that G is a tree.

This shows that the theorem holds for graphs with n vertices. By the induction principle it therefore holds for all positive integers, which was the claim to be shown.

b) Suppose that $G = (V, E)$ is a tree and $u, v \in V$. Suppose there were two different paths from u to v . Then both paths together (after removing eventual common parts of the paths) would form a cycle, contradiction. Vice versa a cycle gives two different paths between two vertices on the cycle.