The bounded topology

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Dedicated to István Juhász
on the occasion of his 80th birthday

Abstract

We introduce a topology on ideals stronger that the usual metric topology as a means for coarse classification of ideals. We study its properties and relation to the combinatorial properties of the ideals. This topology generalizes the submeasure topology on analytic P-ideals introduced by S. Solecki. We give a partial answer to a conjecture of A. Louveau and B. Veličković.

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Introduction

There is an extensive study of combinatorial properties of ideals (see e.g. [7], [9], [10], [17], [18], [19], [23], [24], [25] [26], [29]). Here we propose to use methods of general topology to study combinatorics of (mostly definable) ideals.

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We introduce the *bounded topology* (Definition 1.3) on ideals on ω which is finer than the one inherited by the metric space 2^ω. One of the main features of this topology is that combinatorial notions become topological, e.g. being strongly unbounded is equal to being closed and discrete.

We show that the bounded topology is distinct from the usual metric topology unless the ideal in question is a non-meager P-ideal (Theorem 2.6), and that it coincides for analytic P-ideals with the submeasure topology introduced by S. Solecki in [23] y [24] (Theorem 3.3). In section 2 we study general properties of the bounded topology, combinatorial notions which have their topological counterparts in the bounded topology and topological properties which characterize certain known classes of ideals.Along the way, we identify some topological groups associated with the bounded topology (Theorem 5.4).

This work was largely motivated by a conjecture by A. Louveau and B. Veličković in [19] (Conjecture 4.4), asking whether all ideals which are not the union of countably many weakly bounded sets are Tukey reducible to ω^ω. We discuss the conjecture in section 4 and we present a partial solution (Corollary 4.13) for ideals with a property weaker than being a P-ideal (Definition 4.5 and Proposition 3.4). We also introduce a subideal of $\mathrm{Fin} \times \mathrm{Fin}$, we call the triangular ideal, which does not has this property (Proposition 4.8) and motivates a new conjecture (Conjecture 4.14).

In section 5 we use the bounded topology to classify F_σ-ideals (Theorem 5.11). In section 6 we use a result by T. Banakh and L. Zdomskyĭ to explore when the bounded topology is a topological group. At the end, we present some related questions and conjectures.

1. Preliminaries and terminology

For an infinite set X, a family $\mathcal{I} \subseteq \mathcal{P}(X) \setminus \{X\}$ is an *ideal on* X if it is closed under taking subsets and finite unions. For a set X the notation $[X]^{<\omega}$ stands for $\{A \subseteq X : |A| < \omega\}$ and $[X]^\omega$ stands for $\{A \subseteq X : |A| = \omega\}$. All ideals mentioned in this paper are ideals on ω (or equivalently on a countable set), and always contains the ideal $\mathrm{Fin} = [\omega]^{<\omega}$. The *ideal generated* by a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is the ideal $\langle \mathcal{A} \rangle = \{I \subseteq \omega : (\exists\, \mathcal{F} \in [\mathcal{A}]^{<\omega})\ I \subseteq \bigcup \mathcal{F}\}$. We denote by $\mathcal{I}^+$ the collection of all subsets of ω which do not belong to the ideal $\mathcal{I}$. We can see an ideal as a subset of 2^ω, via the characteristic map, hence it has the induced topology from the metric space 2^ω, which we will denote by τ. The topological concepts that we mention for subsets of an ideal $\mathcal{I}$ (like analytic, open, compact, etc.) refer to the space $(\mathcal{I}, \tau)$, all of them are standard and can be consulted in [5].

For a pair of sets A, B, we use the notation A^B for the set of all functions whose domain is B and their range is contained in A, the notation $A^{<\omega}$ stands for $\bigcup \{A^n : n \in \omega\}$ and $A^{\leq\omega}$ stands for $A^{<\omega} \cup A^\omega$.

For a set $A \subseteq \omega \times \omega$ and $n \in \omega$ let $(A)_n = \{m \in \omega : (n, m) \in A\}$. Throughout the work, we will use the following particular ideals.

- $\mathrm{Fin} \times \emptyset = \{A \subseteq \omega \times \omega : (\forall^\infty\, n \in \omega)\, (A)_n = \emptyset\}$,
- $\emptyset \times \mathrm{Fin} = \{A \subseteq \omega \times \omega : (\exists\, f \in \omega^\omega)\, (\forall\, n \in \omega)\, (A)_n \subseteq f(n)\}$,
- $\mathrm{Fin} \times \mathrm{Fin} = \{A \subseteq \omega \times \omega : (\exists\, f \in \omega^\omega)\, (\forall^\infty\, n \in \omega)\, (A)_n \subseteq f(n)\}$.

An ordered set $(T, \leq)$ is a *tree* if for all $t \in T$, the set $\{s \in T : s \leq t\}$ is well ordered. Trees used here are mainly $(2^{<\omega}, \subseteq)$ and $(\omega^{<\omega}, \subseteq)$. We use standard notation for cones, concatenations and restrictions: for $t \in 2^{<\omega}$, the *cone* determined by t is the set $\langle t \rangle = \{x \in 2^\omega : t \subseteq x\}$, also its *concatenation* with $b \in 2$ is the map $t^\frown b \in 2^{<\omega}$ which extends t in such a way that $\mathrm{dom}(t^\frown b) = \mathrm{dom}(t) \cup \{\mathrm{dom}(t)\}$, $t^\frown b(n) = t(n)$ for all $n \in \mathrm{dom}(t)$ and $t^\frown b(\mathrm{dom}(t)) = b$. For $x \in 2^\omega$ and $n \in \omega$, $x\restriction_n$ denotes the *restriction* of the function x to the set $n = \{0, 1, \ldots, n-1\}$, that is $x\restriction_n(k) = x(k)$ for all $k \in \mathrm{dom}(x\restriction_n) = n$, thus $x\restriction_n \in 2^{<\omega}$. In particular, for $A \subseteq \omega$ the notation $A\restriction_n$ is the restriction of the characteristic map of A to n. We usually take advantage of the identification of the characteristic functions with the sets they represent, and we will use them interchangeably. Similarly, we use the same notation for $\omega^{<\omega}$.

A subset $\mathcal{X} \subseteq \mathcal{I}$ is *bounded* if $\bigcup \mathcal{X} \in \mathcal{I}$. A map $f : \mathcal{I} \to \mathcal{J}$ between a pair of ideals is a *Tukey function* if $f^{-1}[\mathcal{B}] \subseteq \mathcal{I}$ is bounded for every bounded set $\mathcal{B} \subseteq \mathcal{J}$, the existence of such map is denoted by $\mathcal{I} \leq_T \mathcal{J}$. The following is a key notion of a weaker version of bounding and its dual property. These can be defined more generally to directed orders, but we use it only for ideals.

Definition 1.1. Let $\mathcal{I}$ be an ideal.

a) A subset $\mathcal{W} \subseteq \mathcal{I}$ is *weakly bounded* (or a *web set*) if

$$(\forall\, \mathcal{X} \in [\mathcal{W}]^{\omega})\, (\exists\, \mathcal{Y} \in [\mathcal{X}]^{\omega})\ \bigcup \mathcal{Y} \in \mathcal{I}.$$

b) A subset $\mathcal{S} \subseteq \mathcal{I}$ is *strongly unbounded* (or a *sun set*) if

$$(\forall\, \mathcal{X} \in [\mathcal{S}]^{\omega})\ \bigcup \mathcal{X} \notin \mathcal{I}.$$

From its definition, web sets (resp. sun sets) are preserved under almost subsets and finite unions. Furthermore, $\mathcal{W} \subseteq \mathcal{I}$ is a web set if and only if it does not contains any infinite sun set. In this case its *closure under subsets*, i.e. the set $\mathcal{W}^{\downarrow} = \{I \in \mathcal{I} : (\exists\, W \in \mathcal{W})\ \ I \subseteq W\}$, is a web set too.

Proposition 1.2. Let $\mathcal{I}$ be an ideal and $\mathcal{U} \subseteq \mathcal{I}$. The following are equivalent.

i) For all $\mathcal{W} \subseteq \mathcal{I}$ weakly bounded set, $\mathcal{U} \cap \mathcal{W}$ is open in $\mathcal{W}$.

ii) For all $\mathcal{K} \subseteq \mathcal{I}$ weakly bounded and compact, $\mathcal{U} \cap \mathcal{K}$ is open in $\mathcal{K}$.

iii) For all $I \in \mathcal{I}$, $\mathcal{U} \cap \mathcal{P}(I)$ is open in $\mathcal{P}(I)$.

Proof. It follows directly that i) implies ii) implies iii). To see the missing implication, let $\mathcal{W} \subseteq \mathcal{I}$ be a web set and let $I \in \mathcal{U} \cap \mathcal{W}$. If for all $n \in \omega$ there exists $I_n \in (\langle I{\restriction_n}\rangle \cap \mathcal{W}) \setminus \mathcal{U}$, then there is a subsequence $\{I_{n_m} : m \in \omega\}$ that converges to I, which is bounded by some $J \in \mathcal{I}$ and disjoint from $\mathcal{U}$. This, however, contradicts that $\mathcal{U} \cap \mathcal{P}(I \cup J)$ is open in $\mathcal{P}(I \cup J)$. So, there is an $n \in \omega$ such that $\langle I{\restriction_n}\rangle \cap \mathcal{W} \subseteq \mathcal{U}$, and hence $\mathcal{U} \cap \mathcal{W}$ is open in $\mathcal{W}$. ∎

The proposition is clearly true if "open" is replaced by "closed". Also, given a set $\mathcal{U} \subseteq \mathcal{I}$ and a family $\mathcal{X} \subseteq \mathcal{P}(\mathcal{I})$ of web sets that it is $\subseteq$-cofinal among the web sets of $\mathcal{I}$, the set $\mathcal{U}$ satisfies any condition of previous proposition if $\mathcal{U} \cap \mathcal{W}$ is open in $\mathcal{W}$ for all $\mathcal{W} \in \mathcal{X}$. The proposition allows us to define a new topology on ideals which is the main object of study of this work.

Definition 1.3. Let $\mathcal{I}$ be an ideal. Define the *bounded topology on* $\mathcal{I}$, denoted by τ_{bd}, letting $\mathcal{U} \in \tau_{\mathrm{bd}}$ if and only if $\mathcal{U} \subseteq \mathcal{I}$ satisfies any of the conditions in Proposition 1.2.

It follows directly from the definition that τ_{bd} is finer than τ, hence $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a Hausdorff space. As stated before, topological concepts refer to the topology τ, to differentiate between the two topologies we will use the prefix "τ_{bd}-" on properties which refer to the bounded one. For example, we will say that $\mathcal{K} \subseteq \mathcal{I}$ is a τ_{bd}-*compact* if it is compact in the topology τ_{bd}, and we will say that it is a *compact* set if it is compact in the usual topology τ. Also, if $\mathcal{W} \subseteq \mathcal{I}$ is a web set, then there is no difference considering $\mathcal{W}$ as a subspace in the bounded topology or in the usual topology, because in this case both topologies agree.

The following relevant results will be used throughout the paper.

Theorem 1.4 (Louveau and Veličković, [19, Theorem 1 and 2])**.** Let $\mathcal{I}$ be an analytic ideal.

i) $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathcal{I}$ is not the union of less than $\mathfrak{d}$ weakly bounded sets of $\mathcal{I}$.

ii) If $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$, then $\mathcal{I}$ is an F_σ-ideal.

Theorem 1.5 (Jalali-Naini [13] and Talagrand [27])**.** Let $\mathcal{I}$ be an ideal. $\mathcal{I}$ is meager if and only if there exists $\{P_n : n \in \omega\}$, an interval partition of ω, such that $(\forall\, I \in \mathcal{I})\, (\forall^\infty\, n \in \omega)\;\; P_n \not\subseteq I$.

2. Topological and combinatorial results

Proposition 2.1. Let $\mathcal{I}$ be an ideal. Then:

i) $\mathcal{K} \subseteq \mathcal{I}$ is a τ_{bd}-compact set if and only if $\mathcal{K}$ is a compact and weakly bounded.

ii) $\mathcal{S} \subseteq \mathcal{I}$ is a τ_{bd}-closed and τ_{bd}-discrete set if and only if $\mathcal{S}$ is a strongly unbounded.

Proof. To prove the "if" part of i), let $\mathcal{K} \subseteq \mathcal{I}$ be a compact and weakly bounded, and let $\mathfrak{U}$ be an τ_{bd}-open cover of $\mathcal{K}$. By definition of the bounded topology, $\mathcal{V} = \{\mathcal{U} \cap \mathcal{K} : \mathcal{U} \in \mathfrak{U}\}$ is an open cover of $\mathcal{K}$, and since $\mathcal{K}$ is compact, $\mathcal{V}$ has a finite subcover, then $\mathfrak{U}$ has the corresponding finite subcover, hence $\mathcal{K}$ is τ_{bd}-compact.

To prove ii), let $\mathcal{S} \subseteq \mathcal{I}$ be a τ_{bd}-closed and τ_{bd}-discrete set. By the previous paragraph, the set $\mathcal{P}(I) \subseteq \mathcal{I}$ is a τ_{bd}-compact set for all $I \in \mathcal{I}$, so $\mathcal{S} \cap \mathcal{P}(I)$ is finite for all $I \in \mathcal{I}$, hence that $\mathcal{S}$ is a sun set. For the other implication, let $\mathcal{S} \subseteq \mathcal{I}$ be a strongly unbounded set, then for every weakly bounded $\mathcal{W} \subseteq \mathcal{I}$ we have that $\mathcal{S} \cap \mathcal{W}$ is finite, and hence it is closed in $\mathcal{W}$. Therefore $\mathcal{S}$ is a τ_{bd}-closed set in which any of its subsets is also a τ_{bd}-closed set, so it is τ_{bd}-discrete.

To prove the missing part of i) let $\mathcal{K} \subseteq \mathcal{I}$ be a τ_{bd}-compact set, which is a compact set since $\tau \subseteq \tau_{\mathrm{bd}}$. Then $\mathcal{K}$ does not have any infinite τ_{bd}-closed and τ_{bd}-discrete subset, so it does not contain any infinite sun set and therefore it is a web set. ■

So, if a sequence $\mathcal{X} \subseteq \mathcal{I}$ τ_{bd}-converges to some $I \in \mathcal{I}$, it has a bounded subsequence which converges to I; on the other hand, if $\mathcal{X}$ converges to some $I \in \mathcal{I}$ and it is a web set, then $\mathcal{X}$ τ_{bd}-converges to I. This result also implies that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a k-space, later we will show that, actually, it is a sequential space.

As mentioned before, $\mathcal{C} \subseteq \mathcal{I}$ is a τ_{bd}-closed if and only if $\mathcal{C} \cap \mathcal{W}$ is closed in $\mathcal{W}$ for any web set $\mathcal{W} \subseteq \mathcal{I}$. The following result give us another characterization of τ_{bd}-closed sets.

Lemma 2.2. Let $\mathcal{I}$ be an ideal and $\mathcal{F} \subseteq \mathcal{I}$. The following are equivalent.

i) $\mathcal{F}$ is a τ_{bd}-closed set.

ii) If there is a bounded sequence in $\mathcal{F}$ converging to some $I \in \mathcal{I}$, then $I \in \mathcal{F}$.

Proof. To see that i) implies ii), let $\mathcal{X} \subseteq \mathcal{F}$ be a bounded sequence that converges to some $I \in \mathcal{I}$. Then $\mathcal{X} \cup \{I\}$ is weakly bounded. By Proposition 1.2(i), $\mathcal{F} \cap (\mathcal{X} \cup \{I\})$ is a closed set in $\mathcal{X} \cup \{I\}$, therefore $I \in \mathcal{F}$.

For ii) implies i), let $\mathcal{W} \subseteq \mathcal{I}$ be a web set and let $I \in \mathcal{W}$ be a limit point of $\mathcal{F} \cap \mathcal{W}$. Since $\mathcal{W}$ is weakly bounded, there is a bounded sequence $\mathcal{X} \subseteq \mathcal{F}$ which converges to I. Then $I \in \mathcal{F} \cap \mathcal{W}$, and therefore $\mathcal{F} \cap \mathcal{W}$ is a closed set in $\mathcal{W}$. ■

Next we mention some topological properties which hold for any ideal with the bounded topology.

Theorem 2.3. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a homogeneous, separable and sequential space.

Proof. The previous lemma and Proposition 2.1 imply that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is sequential. To prove separability, let $\mathcal{U} \subseteq \mathcal{I}$ be an τ_{bd}-open set and $I \in \mathcal{U}$. Then $\mathcal{U} \cap \mathcal{P}(I)$ is an open set in $\mathcal{P}(I)$ and there is some $n \in \omega$ such that $\langle I\restriction_n \rangle \cap \mathcal{P}(I) \subseteq \mathcal{U} \cap \mathcal{P}(I)$, hence $\mathcal{U}$ contains a finite subset of I. Therefore $[\omega]^{<\omega} \subseteq \mathcal{I}$ is a τ_{bd}-dense set.

Given $I \in \mathcal{I}$, let $\mathrm{trs}_I : \mathcal{I} \to \mathcal{I}$ be the bijection given by $\mathrm{trs}_I(A) = A \triangle I$, i.e. the *translation by* I. For a bounded and convergent sequence $\mathcal{X} \subseteq \mathcal{I}$, $\mathrm{trs}_I(\mathcal{X}) \subseteq \mathcal{I}$ is a bounded and convergent sequence too, hence trs_I is a τ_{bd}-sequentially continuous map, and therefore it is a τ_{bd}-homeomorphism. So, $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homogeneous. ■

The concepts of compactness and sequential compactness coincide in the bounded topology.

Proposition 2.4. Let $\mathcal{I}$ be an ideal and $\mathcal{K} \subseteq \mathcal{I}$. $\mathcal{K}$ is a τ_{bd}-compact set if and only if $\mathcal{K}$ is a τ_{bd}-sequentially compact set.

Proof. Since for metric spaces the concepts of compact and sequentially compact are the same and $(\mathcal{W}, \tau_{\mathrm{bd}}\restriction_{\mathcal{W}}) = (\mathcal{W}, \tau\restriction_{\mathcal{W}})$ for a weakly bounded set $\mathcal{W}$, then it is enough to show that $\mathcal{K} \subseteq \mathcal{I}$ is weakly bounded if it is τ_{bd}-compact or τ_{bd}-sequentally compact. For τ_{bd}-compact set this holds by Proposition 2.1(i). Now let $\mathcal{K} \subseteq \mathcal{I}$ be a τ_{bd}-sequentially compact set, then any sequence of $\mathcal{K}$ has a τ_{bd}-convergent subsequence, in particular, it has a bounded infinite subsequence, therefore $\mathcal{K}$ is weakly bounded. ■

The *Arens space*[3] is the canonical example of a space which is sequential and not Fréchet–Urysohn. In fact, a sequential space is Fréchet–Urysohn if and only if it does not contains a copy of the Arens space see (REF). We will use this to give a characterization of the bounded topology for P-ideals.

Theorem 2.5. Let $\mathcal{I}$ be an ideal. $(\mathcal{I}, \tau_{\mathrm{bd}})$ is Fréchet–Urysohn if and only if $\mathcal{I}$ is a P-ideal.

Proof. We will prove that $\mathcal{I}$ is a non-P-ideal if and only if $(\mathcal{I}, \tau_{\mathrm{bd}})$ contains the Arens space.

Let $\mathcal{F} = \{I_n : n \in \omega\} \subseteq \mathcal{I} \cap [\omega]^\omega$ be a pairwise disjoint family witnessing that $\mathcal{I}$ is not a P-ideal. Increasingly enumerate the set $I_n = \{i^n_m : m \in \omega\}$. For $n, m \in \omega$, let

$$J^n_m = \left\{i^0_n\right\} \cup \bigcup_{k=1}^{n+1} I_k \setminus \left\{i^a_b : 1 \leq a \leq n+1,\ b < m\right\}.$$

We claim that the set $\mathcal{A} = \{J^n_m : n, m \in \omega\} \cup I_0 \cup \{\emptyset\} \subseteq \mathcal{I}$ is homeomorphic to the Arens space. The sequence $\left\{\left\{i^0_n\right\} : n \in \omega\right\}$ is bounded by I_0, so it τ_{bd}-converges to $\emptyset$. Since $\mathcal{F}$ is pairwise disjoint, all points in $\{J^n_m : n, m \in \omega\}$ are isolated. Also, for a fixed n, the sequence $\{J^n_m : m \in \omega\}$ τ_{bd}-converges to $\left\{i^0_n\right\}$ since it is bounded by $\bigcup\{I_k : k \leq n+1\}$. Then it only remains to prove that for every $g \in \omega^\omega$, $\mathcal{X}_g = \{J^n_{g(n)} : n \in \omega\}$ is a strongly unbounded set, because then every diagonal sequence in $\mathcal{A}$ does not τ_{bd}-converge to $\emptyset$. Let $\mathcal{X} \subseteq \mathcal{X}_g$ be an infinite set, since for every $k \in \omega$ there is some $n_k \in \omega$ such that $I_k \subseteq^* J^{n_k}_{g(n_k)}$, then a bound for $\mathcal{X}$ is a pseudo-union for $\mathcal{F}$, therefore $\mathcal{X}_g$ is, indeed, strongly unbounded.

On the other hand, let $\mathcal{A} = \{I^n_m : n, m \in \omega\} \cup \{I_n : n \in \omega\} \cup \{I\}$ be a copy of the Arens space in $(\mathcal{I}, \tau_{\mathrm{bd}})$. For a fixed $n \in \omega$ the sequence $\mathcal{W}_n = \{I^n_m : m \in \omega\}$ is weakly bounded because it τ_{bd}-converges to I_n. By thinning out the space, we

[3]The Arens space can be succinctly defined as the space with the underlying set the ordinal $\omega^2 + 1$ with the strongest topology which makes the sequences $\{n \cdot \omega + k : k \in \omega\}$ and $\{n \cdot \omega\}$ convergent and it is also finer that the order topology.

can suppose that $\mathcal{W}_n$ is actually bounded by some $J_n \in \mathcal{J}$. If $(\forall\, n \in \omega)\ J_n \subseteq^* J$ for some $J \in \mathcal{J}$, then $(\forall\, n \in \omega)\,(\forall^\infty\, m \in \omega)\ I^n_m \subseteq I \cup J$. Hence there are an $X \in [\omega]^\omega$ and a $g \in \omega^\omega$ such that the sequence $\{I^n_{g(n)} : n \in X\}$ τ_{bd}-converges to I. This, however, contradicts the hypothesis on $\mathcal{A}$. Therefore $\{J_n : n \in \omega\}$ witnesses that $\mathcal{J}$ is not a P-ideal. ■

In particular, if the space $(\mathcal{J}, \tau_{\mathrm{bd}})$ is metrizable (or even first-countable) then $\mathcal{J}$ is a P-ideal. So, $\tau = \tau_{\mathrm{bd}}$ is only possible for P-ideals. The following result gives a sufficient and necessary condition for this equality.

Theorem 2.6. Let $\mathcal{J}$ be an ideal. The following are equivalent.

i) $\tau = \tau_{\mathrm{bd}}$.

ii) Any compact subset of $\mathcal{J}$ is weakly bounded.

iii) Any convergent sequence in $\mathcal{J}$ has a bounded subsequence.

iv) $\mathcal{J}$ is a non-meager P-ideal.

Proof. It follows directly that i) implies ii) implies iii).

To see that iii) implies i), let $I \in \mathcal{U}$ for some τ_{bd}-open set $\mathcal{U} \subseteq \mathcal{J}$. If for all n exists $I_n \in (\langle I{\restriction_n}\rangle \cap \mathcal{J}) \setminus \mathcal{U}$ then, by the hypothesis and since $I_n \to I$, the set $\mathcal{K} = \{I_n : k \in \omega\} \cup \{I\} \subseteq \mathcal{J}$ is compact and weakly bounded. Since $\mathcal{U} \cap \mathcal{K} = \{I\}$ then $\mathcal{U}$ is not an τ_{bd}-open set. This proves that $\tau_{\mathrm{bd}} \subseteq \tau$.

For iii) implies iv) note that $\mathcal{J}$ is a non-meager ideal since by Theorem 1.5, any interval partition of ω converges to $\emptyset$. Now, let $\{I_n : n \in \omega\} \subseteq \mathcal{J}$ and for $n \in \omega$ let $J_n = \bigcup_{k \leq n} I_k \setminus n$. Since $\{J_n : n \in \omega\} \subseteq \mathcal{J}$ converges to $\emptyset$, there is a subsequence bounded by some $I \in \mathcal{J}$ which is a pseudo-union of $\{I_n : n \in \omega\}$, so $\mathcal{J}$ is a P-ideal.

iv) implies iii). Let $\{I_n : n \in \omega\} \subseteq \mathcal{J}$ be a sequence which converges to I. Since $\mathcal{J}$ is a P-ideal, there is a $J \in \mathcal{J}$ such that for all $n \in \omega$ the set $F_n = (I_n \setminus I) \setminus J$ is finite. Let $\{E_m : m \in \omega\}$ be an interval partition of ω

such that for all $m \in \omega$, there is some $n_m \in \omega$ with $F_{n_m} \subseteq E_m$, since $\mathcal{I}$ is non-meager. Then there is $A \in [\omega]^\omega$ such that $L = \bigcup \{E_m : m \in A\} \in \mathcal{I}$. Therefore, the subsequence $\{I_{n_m} : m \in A\}$ is bounded by $I \cup J \cup L$. ■

Since Fin is the only ideal which is a strongly unbounded subset of itself, then by Proposition 2.1(ii) the bounded topology is discrete if and only if $\mathcal{I} = \mathrm{Fin}$. In this case, $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homeomorphic to ω and, trivially, there is a set $A \in \mathcal{I}$ such that $\mathcal{P}(A)$ is a τ_{bd}-open set. We know exactly for which ideals this holds.

Lemma 2.7. Let $\mathcal{I}$ be an ideal and let $A \in \mathcal{I}$. Then $\mathcal{P}(A)$ is an τ_{bd}-open set if and only if $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A) = \{I \subseteq \omega : |I \setminus A| < \omega\}$.

Proof. If $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$ then it is straightforward to see that $\mathcal{P}(A)$ is an τ_{bd}-open set since any weakly bounded subset of $\mathcal{I}$ has only finitely many points in its Fin part. On the other hand, let $A \in \mathcal{I}$ such that $\mathcal{P}(A)$ is an τ_{bd}-open set and let $B \in \mathcal{I}$. Since $\mathcal{P}(A) \cap \mathcal{P}(B)$ is an open set in $\mathcal{P}(B)$, there is some $n \in \omega$ such that $\langle (A \cap B)\restriction_n \rangle \cap \mathcal{P}(B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, and then $B \setminus n \in \mathcal{P}(A)$, therefore $B \subseteq^* A$ for all $B \in \mathcal{I}$ which implies that $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$. ■

The lemma implies that the space $(\mathrm{Fin} \oplus \mathcal{P}(A), \tau_{\mathrm{bd}})$ is homeomorphic to $\omega \times 2^\omega$ and, in particular, it is locally compact. It turns out these are the only locally compact ideals in the bounded topology (compare to [23, Corollary 3.2]).

Theorem 2.8. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is locally compact if and only if there is an $A \in \mathcal{I}$ such that $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$.

Proof. Let $\mathcal{I}$ be an ideal such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a locally compact space. Then any point of the space has a τ_{bd}-compact neighborhood which is a metric τ_{bd}-subspace, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ does not contain a copy of Arens space and therefore $\mathcal{I}$ is a P-ideal by Theorem 2.5.

Now, let $\mathcal{U}$ be a τ_{bd}-compact neighborhood of $\emptyset \in \mathcal{I}$. For any $F \in [\omega]^{<\omega}$, let $\mathcal{U}_F$ be the translation of $\mathcal{U}$ by F. Since $[\omega]^{<\omega}$ is a τ_{bd}-dense set and translations are τ_{bd}-homeomorphisms, $\{\mathcal{U}_F : F \in [\omega]^{<\omega}\}$ is a countable family of τ_{bd}-compact sets that covers $\mathcal{I}$. Therefore $\mathcal{I}$ is an F_σ-ideal and it is covered by countable many web sets. Using the theorem Theorem 1.4(i), we conclude that $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$.

Finally, a result due to S. Todorčević (see [28, Section 4]) claims that if $\mathcal{I}$ is an analytic P-ideal, then either $\mathcal{I}$ is countably generated or $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$. By all the previous, we conclude that $\mathcal{I}$ is a P-ideal which is countably generated, therefore there is some $A \in \mathcal{I}$ such that $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$. ∎

We find a necessary condition for the regularity of the bounded topology.

Definition 2.9. An ideal $\mathcal{I}$ has the *shrinking property* if for any pairwise disjoint family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ which is strongly unbounded, there is a strongly unbounded set $\{F_n : n \in \omega\} \subseteq [\omega]^{<\omega}$ of $\mathcal{I}$ such that $(\forall\, n \in \omega)\ F_n \subseteq I_n$.

Proposition 2.10. Let $\mathcal{I}$ be an ideal. If $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a regular space, then $\mathcal{I}$ has the shrinking property.

Proof. Let $\mathcal{I}$ be without the shrinking property and let $\mathcal{S} = \{I_n : n \in \omega\} \subseteq \mathcal{I}$ be a sun set witnessing that. We can assume that $\emptyset \notin \mathcal{S}$. Let $\mathcal{U}$ be an τ_{bd}-open set such that $\mathcal{S} \subseteq \mathcal{U}$. Then $\mathcal{U} \cap \mathcal{P}(I_n)$ is open in $\mathcal{P}(I_n)$ for all $n \in \omega$; therefore there is $F_n \in [\omega]^{<\omega}$ such that $F_n \subseteq I_n$ and $F_n \in \mathcal{U}$. By the hypothesis, and since $F_n \to \emptyset$, there is a bounded subsequence $\{F_{n_k} : k \in \omega\}$ which τ_{bd}-converges to $\emptyset$. Hence $\mathcal{U}$ intersects any τ_{bd}-open neighbourhood of $\emptyset$. So, the τ_{bd}-closed set $\mathcal{S}$ and the point $\emptyset$ proves that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is not regular. ∎

The ideal $\mathrm{Fin} \times \mathrm{Fin}$ does not have the shrinking property. To see this, note that $\mathcal{S} = \{\{n\} \times \omega : n \in \omega\}$ is a sun set and every $\{F_n : n \in \omega\} \subseteq [\omega \times \omega]^{<\omega}$ such that $(\forall\, n)\ F_n \subseteq \{n\} \times \omega$ is a bounded family. Therefore $(\mathrm{Fin} \times \mathrm{Fin}, \tau_{\mathrm{bd}})$ is not a regular space.

If we remove the assumption of disjointness on the hypothesis of Definition 2.9, we have the following stronger property and some related result.

Definition 2.11. An ideal $\mathcal{I}$ has the *strong shrinking property* if for any family $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ which is a strongly unbounded set of $\mathcal{I}$, there is a strongly unbounded set $\{F_n : n \in \omega\} \subseteq [\omega]^{<\omega}$ of $\mathcal{I}$ such that $(\forall\, n \in \omega)\ F_n \subseteq I_n$.

Using the later Lemma 5.1, if $\mathcal{I}$ is an ideal such that the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is σ-compact then $\mathcal{I}$ has the strong shrinking property. For if $\{\mathcal{K}_n : n \in \omega\}$ is the family given by the lemma and $\mathcal{S} = \{S_n : n \in \omega\}$ is a sun set of $\mathcal{I}$; then choose $F_0 \subseteq S_0$ finite and assuming $F_i \subseteq S_i$ have been chosen for $i \leq n$, note that $\mathcal{K}_{n+1}$ can only contain finitely many elements from $\mathcal{S}$, for those S_k such that $k > n$ and $S_k \in \mathcal{K}_{n+1}$ choose a finite $F_k \in \mathcal{P}(S_k) \cap (\mathcal{K}_{n+1} \setminus \mathcal{K}_n)$. Then the sequence $\{F_n : n \in \omega\}$ is necessarily a sun subset.

Proposition 2.12. Let $\mathcal{I}$ be an ideal. If $\mathcal{I}$ has a perfect strongly unbounded subset, then $\mathcal{I}$ does not have the strong shrinking property.

Proof. Let $\mathcal{P} \subseteq \mathcal{I}$ be a perfect strongly unbounded set, and $\mathcal{D} \subseteq \mathcal{P}$ be a countable dense subset. For all $x \in \mathcal{D}$, let $F_x \in [x]^{<\omega}$. We recursively define a sequence $\langle x_n : n \in \omega \rangle \subseteq \mathcal{D}$ as follows: let $x_0 \in \mathcal{D}$, if x_n is already defined, then let $x_{n+1} \in \mathcal{D}$ such that $\min\{k : x_n(k) \neq x_{n+1}(k)\} > \max F_{x_n}$. Since $\mathcal{P}$ is perfect, we can assume that $\langle x_n : k \in \omega \rangle$ converges to some $x^* \in P$, hence $(\forall\, n \in \omega)\, F_{x_n} \subseteq x^*$ and therefore $\{F_x : x \in \mathcal{D}\}$ is not a sun set. Thus $\mathcal{D}$ witnesses that $\mathcal{I}$ does not have the strong shrinking property. ■

In order to give a sufficient condition for the regularity of the bounded topology, we will use a concept given by K. Kawamura, L. Oversteegen and E. Tymchatyn in [15, Definition 1]. A topological space (X, τ_1) is *almost zero-dimensional* if there is a topology τ_0 coarser than τ_1 such that (X, τ_0) is a zero-dimensional space and every point in X has a τ_1-neighborhood basis consisting of τ_0-closed sets. For $(\mathcal{I}, \tau_{\mathrm{bd}})$, the usual topology seems to be the natural witness for that property, but this does not always hold since any almost zero-dimensional space is regular. Then we have the following question.

Question 2.13. If $(\mathcal{I}, \tau_{\mathrm{bd}})$ is an almost zero-dimensional space, then it is regular, which in turn implies that $\mathcal{I}$ has the shrinking property. Which of these implications are reversible?

3. Analytic P-ideals

Definition 3.1. A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if it is

- [proper] $(\forall\, n \in \omega)\ \ \varphi(\{n\}) < \infty$ and $\varphi(\emptyset) = 0$.
- [monotone] $(\forall\, A, B \subseteq \omega)\ \ A \subseteq B \ \rightarrow\ \varphi(A) \leq \varphi(B)$
- [subadditive] $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$

If additionally $(\forall\, A \subseteq \omega)\ \ \varphi(A) = \lim_n \varphi(A \cap n)$, then φ is a *lower semicontinuous submeasure.*

One of the main results about analytic P-ideal is due to S. Solecki (see [23]), it says that $\mathcal{I}$ is an analytic P-ideal if and only if there exists a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathrm{Exh}\,(\varphi) = \{X \subseteq \omega : \lim_{n\to\infty} \varphi(X \setminus n) = 0\}$. In this section we prove that the topology τ_{bd} coincides with the associated topology given by the Solecki's Theorem.

Lemma 3.2. Let φ be a lower semicontinuous submeasure and $\mathcal{W} \subseteq \mathrm{Exh}\,(\varphi)$. Then $\mathcal{W}$ is weakly bounded if and only if it satisfies

$$(\forall\, \varepsilon > 0)\,(\exists\, N \in \omega)\,(\forall\, I \in \mathcal{W})\ \ \varphi(I \setminus N) < \varepsilon. \tag{$*$}$$

Proof. Let $\mathcal{W} \subseteq \mathrm{Exh}\,(\varphi)$ and $\varepsilon > 0$ with $(\forall\, N \in \omega)\,(\exists\, I \in \mathcal{W})\ \ \varphi(I \setminus N) \geq 2\varepsilon$. Then there is an increasing sequence $\{N_m : m \in \omega\}$ and a family $\mathcal{S} = \{I_m : m \in \omega\} \subseteq \mathcal{W}$ such that $\varphi\,(I_m \cap N_{m+1} \setminus N_m) \geq \varepsilon$. Hence for an infinite $\mathcal{X} \subseteq \mathcal{S}$ we have $(\forall\, N \in \omega)\ \ \varphi\,(\bigcup \mathcal{X} \setminus N) \geq \varepsilon$, and then $\mathcal{S}$ is a sun set. Therefore any web set must satisfy ($*$).

Now, let $\mathcal{W} \subseteq \mathrm{Exh}\,(\varphi)$ be an infinite set satisfying ($*$) and $\mathcal{X} = \{I_n : n \in \omega\} \subseteq \mathcal{W}$. Without loss of generality, we can assume that $\mathcal{X}$ converges to I for some $I \in \mathrm{Exh}\,(\varphi)$. For all $m \in \omega$ we increasingly choose a number N_m as a witness of ($*$) for $\varepsilon_m = \frac{1}{2^m(m+1)}$ such that there is $I_{n_m} \in \mathcal{X}$ with $\min\{k \in \omega : I_{n_m}(k) \neq I(k)\} \in [N_m, N_{m+1})$. We have that $J = \bigcup\{I_{n_m} : m \geq 1\} \in \mathrm{Exh}\,(\varphi)$ since $\varphi(J \setminus N_m) \leq \sum_{k\geq m} \frac{1}{2^k}$, then $\mathcal{X}$ has a bounded subsequence and therefore $\mathcal{W}$ is a web set. ∎

For a lower semicontinuous submeasure φ, we denote by τ_φ to the topology on the ideal $\mathrm{Exh}\,(\varphi)$ induced by the metric d_φ given by $d_\varphi(I, J) = \varphi(I \triangle J)$.

Theorem 3.3. Let $\mathcal{I}$ be an analytic P-ideal and φ the associate lower semicontinuous submeasure. Then $\tau_{\mathrm{bd}} = \tau_\varphi$.

Proof. For $\varepsilon > 0$, we define $B^{\varphi}_{\varepsilon} = \{J \in \mathcal{I} : \varphi(Y) < \varepsilon\}$.

To see that $\tau_{\varphi} \subseteq \tau_{\mathrm{bd}}$ is enough to prove that for all $\varepsilon > 0$, $B^{\varphi}_{\varepsilon}$ is an τ_{bd}-open set. Let $\mathcal{K} \subseteq \mathcal{I}$ be a compact and web set, and let $I \in B^{\varphi}_{\varepsilon} \cap \mathcal{K}$. If for all $n \in \omega$ there is $I_n \in (\langle I{\restriction_n}\rangle \cap \mathcal{K}) \setminus B^{\varphi}_{\varepsilon}$, then $\{I_n : n \in \omega\} \subseteq \mathcal{K}$ is an infinite sun set, which contradicts that $\mathcal{K}$ is a web set. Hence, there is some $n \in \omega$ such that $I \in \langle I{\restriction_n}\rangle \cap \mathcal{K} \subseteq B^{\varphi}_{\varepsilon}$, and therefore $B^{\varphi}_{\varepsilon}$ is an τ_{bd}-open set.

On the other hand, let $\mathcal{U}$ be an τ_{bd}-open neighborhood of $\emptyset$. It is enough to prove that there is some $\varepsilon > 0$ such that $B^{\varphi}_{\varepsilon} \subseteq \mathcal{U}$. If it is not the case, for all $n \in \varepsilon$ there is some $I_n \in B^{\varphi}_{\varepsilon} \setminus \mathcal{U}$. By the previous lemma $\mathcal{K} = \{I_n : n \in \omega\} \cup \{\emptyset\}$ is a compact and web set, but this contradicts that $\mathcal{U}$ is a τ_{bd}-open set since $\mathcal{U} \cap \mathcal{K} = \{\emptyset\}$. ∎

It is no difficult to see that for an analytic P-ideal $\mathcal{I}$ the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is almost zero-dimensional with respect to the usual topology, because for all $r > 0$ the set $\{A \subseteq \omega : \varphi(A) \leq r\} \subseteq \mathcal{I}$ is closed in $\mathcal{I}$ since φ is semicontinuous. As will be shown in Theorem 5.11, the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is not zero-dimensional if $\mathcal{I}$ is a P-ideal which is F_{σ} and $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$.

Proposition 3.4. Let $\mathcal{I}$ be an analytic P-ideal on ω. Then the closure of any weakly bounded set of $\mathcal{I}$ is also a weakly bounded set of $\mathcal{I}$.

Proof. Let φ a lower semicontinuous submeasure with $\mathcal{I} = \mathrm{Exh}\,(\varphi)$. Let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, its straightforward to see that $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{I}$ by the properties of φ. Let $\mathcal{X} \subseteq \mathrm{cl}(\mathcal{W})$ an infinite set, then there is a sequence $\{I_n : n \in \omega\} \subseteq \mathcal{X}$ which converges to I for some $I \in \mathcal{I}$.

We claim that $\lim_n \varphi(I_n \setminus I) = 0$. If not, there exist $\varepsilon > 0$ and a subsequence $\{I_{n_k} : k \in \omega\}$ such that $(\forall\, k \in \omega)\ \varphi((I_{n_k} \setminus I) \cap m_k) > \varepsilon$, where $m_k = \min(I_{n_{k}+1} \setminus I)$. Since $I_{n_k} \in \mathrm{cl}(\mathcal{W})$, for all $k \in \omega$ there is some $W_k \in \mathcal{W}$ such that $W_k \cap m_k = I_{n_k} \cap m_k$. This implies that $\{W_k : k \in \omega\} \subseteq \mathcal{W}$ is a sun set; thus the claim holds. Finally, the previous lemma implies that the sequence $\{I_n : n \in \omega\}$ is a web set and therefore $\mathrm{cl}(\mathcal{W})$ is a web set too. ∎

The property in the conclusion of Proposition 3.4 will be relevant in the following section. We know that for a P-ideal, if it is analytic or non-meager, then its bounded topology is metrizable, we have the following question

Question 3.5. Is there a P-ideal $\mathcal{I}$ such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is not a metric space?

4. The conjecture of Louveau and Veličković

Definition 4.1. Let $\mathcal{I}$ be an ideal. The *weakly bounded number* of $\mathcal{I}$ is defined as

$$\mathrm{web}(\mathcal{I}) = \min\left\{|\mathcal{X}| : \mathcal{X} \subseteq \mathcal{P}(\mathcal{I}),\ (\forall \mathcal{W} \in \mathcal{X})\, \mathcal{W} \text{ is a web set, and } \bigcup \mathcal{X} = \mathcal{I}\right\}.$$

From the definition it follows directly that $\mathrm{web}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$ for any ideal $\mathcal{I}$. It is straightforward to see that for any sun set $\mathcal{S} \subseteq \mathcal{I}$, $|\mathcal{S}| \leq \mathrm{web}(\mathcal{I})$. The following is a folklore result.

Proposition 4.2. Let $\mathcal{I}, \mathcal{J}$ be ideals such that $\mathcal{I} \leq_T \mathcal{J}$. The following holds.

i) $\mathrm{web}(\mathcal{I}) \leq \mathrm{web}(\mathcal{J})$.

ii) Let $\mathcal{S} \subseteq \mathcal{I}$ be a sun set, then there is a sun set $\mathcal{S}' \subseteq \mathcal{J}$ such that $|\mathcal{S}| = |\mathcal{S}'|$.

The *extent* of a topological space (X, τ), denoted by $e(X, \tau)$, is the supremum of the size of closed discrete subspaces of X. Hence, $\omega \leq e(\mathcal{I}, \tau_{\mathrm{bd}}) \leq \mathrm{web}(\mathcal{I})$ since the set of all initial segments of ω is a sun set of any ideal $\mathcal{I}$.

K. Beros and P. Larson showed that the *summable ideal*, defined as $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum \left\{n^{-1} : n \in A \setminus \{0\}\right\} \text{ converges}\right\}$, has no uncountable sun sets (see [2, Proposition 3.4]). Also, S. Todorčević showed in [28] that $\mathcal{I} \leq_T \mathcal{I}_{1/n}$ for any analytic P-ideal $\mathcal{I}$. Then, by Proposition 4.2(ii), in this class of ideals we have that $e(\mathcal{I}, \tau_{\mathrm{bd}}) = \omega$.

If any web set of an ideal $\mathcal{I}$ is bounded, then $\mathrm{web}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$. Hence, using the following result, we deduce that $\mathrm{web}(\emptyset \times \mathrm{Fin}) = \mathfrak{d}$.

Proposition 4.3. Any weakly bounded subset of $\emptyset \times \mathrm{Fin}$ is bounded.

Proof. Let $\mathcal{B} \subseteq \emptyset \times \mathrm{Fin}$ be an unbounded set, then $\bigcup \mathcal{B} \cap \{m\} \times \omega$ is infinite for some $m \in \omega$. Therefore, for any $n \in \omega$ exists $B_n \in \mathcal{B}$ such that $\max\{k : (m, k) \in B_n\} \geq n$. Hence the set $\{B_n : n \in \omega\} \subseteq \mathcal{B}$ is a sun set, thus $\mathcal{B}$ is not a web set. ∎

Using the previous and our notation, Theorem 1.4 says that if $\mathcal{I}$ is an F_σ-ideal then $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathrm{web}(\mathcal{I}) \geq \mathfrak{d}$. A. Louveau and B. Veličković conjectured that this result can be improved.

Conjecture 4.4 (Louveau and Veličković, [19, Conjecture 1])**.** Let $\mathcal{I}$ be an F_σ-ideal, then $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathrm{web}(\mathcal{I}) > \omega$.

We will prove that this holds for the following class of ideals.

Definition 4.5. An ideal $\mathcal{I}$ has the *web-closure property* if the closure of any weakly bounded set of $\mathcal{I}$ is a weakly bounded set of $\mathcal{I}$.

Note that the closure of a web set seen in 2^ω is the same that its closure seen in the ideal $\mathcal{I}$. Indeed, let $\mathcal{W} \subseteq \mathcal{I}$ be a web set and $I \in \mathrm{cl}(\mathcal{W})$, then there is a sequence $\{W_n : n \in \omega\} \subseteq \mathcal{W}$ which converges to I. Since $\mathcal{W}$ is a web set, then there is a bounded subsequence $\{W_{n_k} : k \in \omega\}$ which also converges to I, then $I \subseteq \bigcup_{k\in\omega} W_{n_k} \in \mathcal{I}$, therefore $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{I}$. This helps to prove the following.

Proposition 4.6. Let $\mathcal{I}$ be an ideal. Then $\mathcal{I}$ has the web-closure property if and only if any weakly bounded set of $\mathcal{I}$ is contained in some τ_{bd}-compact set of $\mathcal{I}$.

Proof. To see the "only if" part, let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, by hypothesis $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{I}$ is a web set, and since it is a closed set of 2^ω, then it is compact. Therefore, $\mathrm{cl}(\mathcal{W})$ is a τ_{bd}-compact set and $\mathcal{W}$ is contained in it. On the other hand, let $\mathcal{W} \subseteq \mathcal{I}$ be a web set, there is some τ_{bd}-compact set $\mathcal{K}$ such that $\mathcal{W} \subseteq \mathcal{K}$. Since $\mathcal{K}$ is compact, then $\mathrm{cl}(\mathcal{W}) \subseteq \mathcal{K}$, and since $\mathcal{K}$ is a web set, then $\mathrm{cl}(\mathcal{W})$ is also a web set. Therefore, $\mathcal{I}$ has the web-closure property. ■

Corollary 4.7. Let $\mathcal{I}$ be an ideal with the web-closure property. Then $\mathrm{web}(\mathcal{I})$ is the minimum size of a family of τ_{bd}-compact sets which covers $\mathcal{I}$.

Proof. Let $\mathfrak{K}(\mathcal{I}) = \min\{|\mathfrak{X}| : (\forall \mathcal{K} \in \mathfrak{X})\ \mathcal{K} \text{ is a } \tau_{\mathrm{bd}}\text{-compact set, and } \bigcup \mathfrak{X} = \mathcal{I}\}$. Since any τ_{bd}-compact set is a web set, we always have that $\mathrm{web}(\mathcal{I}) \leq \mathfrak{K}(\mathcal{I})$. Moreover, since $\mathcal{I}$ has the web-closure property, then any web set is contained in some τ_{bd}-compact set. Hence a witness for $\mathrm{web}(\mathcal{I})$ give us a witness for $\mathfrak{K}(\mathcal{I})$, and therefore these are equal. ■

Not every ideal has the web-closure property. Let $T \subseteq \omega\times\omega$, we say that T is *infinite-triangular* if $T = \bigcup_{k\in\omega} \{n_k\} \times (n_{k+1} - n_k)$ for some increasing sequence $\{n_k : k \in \omega\}$; and we say that T is *finite-triangular* if there is an increasing finite sequence $\{n_0, \ldots, n_m\}$ such that $T = \left(\bigcup_{k<m} \{n_k\} \times (n_{k+1} - n_k)\right) \cup \{n_m\} \times \omega$. Note that $\emptyset$ is finite-triangular by the empty sequence. A set $T \subseteq \omega \times \omega$ is *triangular* if it is either finite-triangular or infinite-triangular. Then, we define the *triangular ideal* $\mathcal{I}_{\mathcal{T}}$ as follows

$$\mathcal{I}_{\mathcal{T}} = \langle \{T \subseteq \omega \times \omega : T \text{ is a triangular set}\}\rangle.$$

Note that $\mathcal{I}_{\mathcal{T}} \subseteq \mathrm{Fin} \times \mathrm{Fin}$.

Proposition 4.8. $\mathcal{I}_{\mathcal{T}}$ is a tall F_σ-ideal which does not have neither the shrinking property nor the web-closure property and it has a sun set of size $\mathfrak{c}$.

Proof. For $A \subseteq \omega$ and $n \in \omega$ let $A(n)$ be the n-th element of A, if exists. Then $\mathcal{P}(\omega)$ is in correspondence with the set of triangular sets via the map $\mathbb{T} : \mathcal{P}(\omega) \to \omega \times \omega$ given by $\mathbb{T}(A) = \bigcup_{n\in\omega} \{A(n)\} \times (A(n+1) - A(n))$ if A is infinite, $\mathbb{T}(A) = \left(\bigcup_{n=0}^{k} \{A(n)\} \times (A(n+1) - A(n))\right) \cup \{A(k+1)\} \times \omega$ if $|A| = k+1$ and $\mathbb{T}(\emptyset) = \emptyset$. Since $\mathbb{T}$ is a continuous map, the set of triangular sets is compact, and therefore $\mathcal{I}_{\mathcal{T}}$ is an F_σ-ideal.

The definition of a triangular set directly implies that $\mathcal{I}_{\mathcal{T}}$ is a tall ideal. Therefore, the set $\mathcal{W} = \{\{n\} \times m : n, m \in \omega\} \subseteq \mathcal{I}_{\mathcal{T}}$ is a web set. Now, for all $n \in \omega$, the set $\{n\}\times\omega$ belongs to the closure of $\mathcal{W}$, and since $\{\{n\} \times \omega : n \in \omega\} \subseteq \mathcal{I}_{\mathcal{T}}$ is a sun set, the ideal $\mathcal{I}_{\mathcal{T}}$ does not has the web-closure property.

Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family of size $\mathfrak{c}$. Let $A_0, \ldots, A_n$ be $n \geq 1$ distinct elements of $\mathcal{A}$, then there is some $N \in \omega$ such that the family $\{A_i \setminus N : i \leq n\}$ is pairwise disjoint, therefore exists $m_0, \ldots, m_n \geq N$ such that $(\forall\, k < n)\ \ A_k(m_k) < A_n(m_n) < A_k(m_k + 1)$ and, reindexing if necessary, also $(\forall\, i < j < n)\ A_i(m_i) < A_j(m_j)$. For $k \leq n$, let $B_k = \{A_k(m_k)\}\times(A_k(m_k+1) - A_k(m_k))$, by previous, if a triangular set covers B_k, then it cannot cover any B_l for $k < l \leq n$, this implies that $\bigcup_{k\leq n} \mathbb{T}(A_k)$ cannot be covered by n triangular sets. Hence any infinite subset of $\mathcal{S} = \{\mathbb{T}(A) : A \in \mathcal{A}\} \subseteq \mathcal{I}_{\mathcal{T}}$ is unbounded, therefore $\mathcal{S}$ is a sun set of size $\mathfrak{c}$.

Finally, let $\{A_n : n \in \omega\} \subseteq \mathcal{P}(\omega)$ be a pairwise disjoint family such that $(\forall\, n \in \omega)\,(\forall\, k < n)\;\; A_n(0) > A_k(n-k)$. By the previous paragraph, we have that $\{\mathbb{T}(A_n) : n \in \omega\}$ is a sun set. Let $F_n \in [\mathbb{T}(A_n)]^{<\omega}$ for all $n \in \omega$. Recursively define sequences $\{n_k : k \in \omega\}$, $\{m_k : k \in \omega\} \subseteq \omega$ as follows. Start with $n_0 = 0$. Suppose n_k is already defined, since F_{n_k} is finite, let m_k such that $F_{n_k} \subseteq \mathbb{T}(A_{n_k}) \cap A_{n_k}(m_k) \times \omega$ and choose n_{k+1} such that $A_{n_{k+1}}(0) > A_{n_k}(m_k)$. Then $I = \bigcup \{\mathbb{T}(A_{n_k}) \cap A_{n_k}(m_k) \times \omega : k \in \omega\} \in \mathcal{I}_{\mathcal{T}}$ and hence $\{F_n : n \in \omega\}$ has a bounded infinite subset, therefore $\mathcal{I}$ has no the shrinking property. ■

By Proposition 3.4 we know that any P-ideal is in the class of ideals with the web-closure property; nevertheless, these classes are not equal. A witness for that is the *polynomial growth ideal*, introduced in [19, Example 1] and defined by the following

$$\mathcal{I}_{\mathcal{P}} = \left\{A \subseteq \omega : (\exists\, k \in \omega)\,(\forall\, n \in \omega)\;\; |A \cap 2^n| \leq n^k\right\}.$$

It is tall, not countably generated, F_σ and not a P-ideal. Then $\omega = \mathrm{web}(\mathcal{I}_{\mathcal{P}}) < \mathrm{cof}\,(\mathcal{I}_{\mathcal{P}})$ since for all $k \in \omega$, $\mathcal{W}_k = \left\{A \subseteq \omega : (\forall\, n \in \omega)\;\; |A \cap 2^n| \leq n^k\right\} \subseteq \mathcal{I}_{\mathcal{P}}$ is a τ_{bd}-compact set and the family $\{\mathcal{W}_k : k \in \omega\}$ is a cover for $\mathcal{I}_{\mathcal{P}}$. Using Theorem 5.2, we know that this ideal has the web-closure property.

To prove that any ideal with the web-closure property satisfies the conjecture, we need some previous lemmas.

Lemma 4.9. Let $\mathcal{I}$ be a meager ideal and let $\mathcal{W} \subseteq \mathcal{I}$ be a weakly bounded set. Then $\mathcal{W}$ is nowhere dense in $\mathcal{I}$.

Proof. Let $\{P_n : n \in \omega\}$ be the interval partition of ω given by Theorem 1.5. Suppose that $\mathcal{W}$ is dense in $\mathcal{I}$ above some $s \in 2^{<\omega}$. Then, there is an increasing sequence $\{s_k : k \in \omega\} \subseteq 2^{<\omega}$ such that $s_0 = s$ and for every $k \geq 1$ exists $n_k \in \omega$ satisfying $(\forall\, m \in P_{n_k})\, s_k(m) = 1$. Hence we can choose a subset $\mathcal{S} = \{S_k : k \in \omega\} \subseteq \mathcal{W}$ satisfying $(\forall\, k \in \omega)\;\; P_{n_k} \subseteq S_k$. Thus, due the property of the partition, $\mathcal{S}$ is a sun set of $\mathcal{I}$, a contradiction. Therefore, $\mathcal{W}$ is a nowhere dense set on the ideal $\mathcal{I}$. ■

In what follows, if $a, b \in [\omega]^{<\omega}$ we use the notation $a \sqsubseteq b$ if $a \subseteq b$ and $(\forall\, n \in a)\, (\forall\, m \in b)\;\; m \leq n \;\rightarrow\; m \in a$.

Definition 4.10. Let $\mathcal{I}$ be an ideal. A family $\mathcal{A} = \{a_s : s \in \omega^{<\omega}\} \subseteq [\omega]^{<\omega}$ is a *sun-branching tree on* $\mathcal{I}$ if it satisfies the following.

- $a_\emptyset = \emptyset$.
- $(\forall\, s \in \omega^{<\omega})\, (\forall\, n \in \omega)\;\; a_s \sqsubseteq a_{s^\frown n}$.
- $(\forall\, s \in \omega^{<\omega})\;\; \{a_{s^\frown n} : n \in \omega\}$ is a sun set of $\mathcal{I}$.
- $(\forall\, x \in \omega^\omega)\;\; \bigcup\{a_{x\restriction_n} : n \in \omega\} \in \mathcal{I}$.

A family $\mathcal{F} \subseteq [\omega]^{<\omega}$ is a *finite-branching tree on* $\mathcal{I}$ if it satisfies all previous conditions but the third one replacing "sun" by "finite"; that is, only finitely many of the sets $a_{s^\frown n}$ are different.

Lemma 4.11. Let $\mathcal{I}$ be an ideal. If there is a sun-branching tree on $\mathcal{I}$, then $\mathrm{web}(\mathcal{I}) \geq \mathfrak{d}$.

Proof. Let $\mathcal{A} = \{a_s : s \in \omega^{<\omega}\} \subseteq [\omega]^{<\omega}$ be a sun-branching tree on $\mathcal{I}$. For $\mathcal{X} \subseteq \mathcal{A}$ let

$$[\mathcal{X}]_\infty = \left\{\bigcup\left\{a_{x\restriction_n} : n \in \omega\right\} : x \in \omega^\omega \text{ and } (\exists^\infty\, n \in \omega)\;\; a_{x\restriction_n} \in \mathcal{X}\right\} \subseteq \mathcal{I}.$$

Thus, $[\mathcal{X}]_\infty$ is the collection of elements in the ideal $\mathcal{I}$ which have infinitely many initial segments in a branch of $\mathcal{X}$. We will prove that for any web set $\mathcal{W} \subseteq \mathcal{I}$ exists $\mathcal{F} \subseteq \mathcal{A}$, a finite-branching tree on $\mathcal{I}$, such that $\mathcal{W} \cap [\mathcal{A}]_\infty \subseteq [\mathcal{F}]_\infty$, which shows that $\mathrm{web}(\mathcal{I}) \geq \mathfrak{d}$ holds because $\mathfrak{d}$ many sets of branches from finite-branching subtrees of the tree $\omega^{<\omega}$ are needed to cover the space ω^ω.

We can assume that $\mathcal{W} = \mathcal{W}^\downarrow$. For all $s \in \omega^{<\omega}$ let $F_s = \mathcal{W} \cap \{a_{s^\frown n} : n \in \omega\}$, then $\mathcal{F} = \bigcup\{F_s : s \in \omega^{<\omega}\} \subseteq \mathcal{A}$ is a finite-branching tree on $\mathcal{I}$. Let $I \in \mathcal{W} \cap [\mathcal{A}]_\infty$, there is $x \in \omega^\omega$ such that $(\forall\, n \in \omega)\;\; a_{x\restriction_{n+1}} \in \mathcal{A} \cap \mathcal{W}$ and $I = \bigcup\left\{a_{x\restriction_n} : n \in \omega\right\}$, then $a_{x\restriction_{n+1}} \in F_{x\restriction_n}$ for all $n \in \omega$, and therefore $I \in [\mathcal{F}]_\infty$. ■

Theorem 4.12. Let $\mathcal{I}$ be an F_σ-ideal with the web-closure property. Then either $\mathrm{web}(\mathcal{I}) = \omega$ or $\mathrm{web}(\mathcal{I}) \geq \mathfrak{d}$.

Proof. We define an infinite game with perfect information $\mathfrak{G}_{web}(\mathcal{I})$ as follows.

$$\begin{array}{c|ccccccccc}
\mathrm{I} & \mathcal{W}_0 & & \mathcal{W}_1 & & \cdots & & \mathcal{W}_n & & \cdots \\
\hline
\mathrm{II} & & a_0 & & a_1 & & \cdots & & a_n & & \cdots
\end{array}$$

At the nth move, Player I chooses a web set $\mathcal{W}_n$ of $\mathcal{I}$ such that $\mathcal{W}_n = \overline{\mathcal{W}_n^{\downarrow}}$, and Player II chooses a finite subset a_n such that $a_n \notin \mathcal{W}_n$ and $a_n \sqsubseteq a_{n+1}$ for all n, this is possible by Lemma 4.9. Player II wins a run of the game if and only if $\bigcup \{a_n : n \in \omega\} \in \mathcal{I}$. Since $\mathcal{I}$ is Borel then $\mathfrak{G}_{web}(\mathcal{I})$ is determined due to Martin's Determinacy Theorem for Borel games. So, we will consider the two cases.

Case 1. Player I has a winning strategy, say σ. Since in every move Player II has countably many options to choose, σ determines a countable family of web sets of $\mathcal{I}$, namely $\mathcal{X}$, which consists of all responses of Player I in σ. Now, if there exists $I \in \mathcal{I} \setminus \bigcup \mathcal{X}$ then there is a run of the game in which Player II choose an initial segment of I in every move since $\mathcal{W} = \overline{\mathcal{W}^{\downarrow}}$ for all $\mathcal{W} \in \mathcal{X}$, thus Player II would win the run, which is not possible by σ. Then $\bigcup \mathcal{W} = \mathcal{I}$ and $\mathrm{web}(\mathcal{I}) = \omega$.

Case 2. Player II has a winning strategy, say λ. We will prove that exists a sun-branching tree on $\mathcal{I}$. Let $\mathcal{X}_t = \left\{a \in [\omega]^{<\omega} : (\exists\, \mathcal{W} \subseteq \mathcal{I} \text{ web set})\ \ t^\frown(\mathcal{W}, a) \in \lambda\right\}$ for every $t \in \lambda$ of even length, $\mathcal{X}_t$ cannot be a web set of $\mathcal{I}$ because Player II could not do his next move if Player I chooses $\overline{\mathcal{X}_t^{\downarrow}}$ as his move. Then, for every suitable $t \in \lambda$, let $\mathcal{S}_t \subseteq \mathcal{X}_t$ be a countable sun set of $\mathcal{I}$, and for every $a \in \mathcal{S}_t$ let $\mathcal{W}_a^t$ be a web set such that $t^\frown(\mathcal{W}_a^t, a) \in \lambda$. Let $N_t = \left\{t^\frown(\mathcal{W}_a^t, a) \in \lambda : a \in \mathcal{S}_t\right\}$. Recursively define the sequence $\{M_n \in \mathcal{P}(\lambda) : n \in \omega\}$ given by $M_0 = \{\emptyset\}$ and $M_{n+1} = \bigcup \{N_t : t \in M_n\}$. Finally, $\mathcal{A} = \{\emptyset\} \cup \bigcup_{n\in\omega} \bigcup_{t \in M_n} \mathcal{S}_t$ is a sun-branching tree on $\mathcal{I}$ since every $\mathcal{S}_t$ is a sun set and λ is a winning strategy for Player II. ■

By the previous result, the fact that $\mathrm{web}(\emptyset \times \mathrm{Fin}) = \mathfrak{d}$, Theorem 1.4 and Proposition 4.2 we conclude the following.

Corollary 4.13. Let $\mathcal{I}$ be an analytic ideal with the web-closure property. Then $\emptyset \times \mathrm{Fin} \leq_T \mathcal{I}$ if and only if $\mathrm{web}(\mathcal{I}) > \omega$.

We have the following strengthening of Conjecture 4.4.

Conjecture 4.14. Let $\mathcal{I}$ be an F_σ-ideal. Then either $\mathcal{I}$ has the web-closure property or $\mathcal{I}$ has a strongly unbounded set of size $\mathfrak{c}$.

It is perhaps worth mentioning that that this consistently fails for analytic ideals by a result of T. Mátrai (see [20, Corollary 5.22])

5. $\mathbf{F}_\sigma$-ideals

If $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space, then $\mathcal{I}$ is an F_σ-ideal, since any τ_{bd}-compact set is, in particular, a closed set. The following is a useful lemma for this topological property.

Lemma 5.1. Let $\mathcal{I}$ be an ideal. If there is an increasing countable family of τ_{bd}-compact sets $\mathcal{K}$ which covers the ideal and $\mathcal{K} = \mathcal{K}^{\downarrow}$ for all $\mathcal{K} \in \mathcal{K}$, then $\mathcal{K}$ is cofinal among the weakly bounded sets of $\mathcal{I}$. Furthermore, if $(\mathcal{I}, \tau_{\mathrm{bd}})$ is σ-compact such family exists.

Proof. For the first part, let $\mathcal{K} = \{\mathcal{K}_n : n \in \omega\}$ be such a family. Let $\mathcal{S} \subseteq \mathcal{I}$ such that for all $n \in \omega$ there exists $I_n \in \mathcal{S} \setminus \mathcal{K}_n$. Let $\mathcal{X} \subseteq \{I_n : n \in A\}$ be a bounded set. There is $m \in \omega$ such that $\bigcup \mathcal{X} \in \mathcal{K}_m = \mathcal{K}_m^{\downarrow}$ and hence $\mathcal{X} \subseteq \mathcal{K}_m$, thus $\mathcal{X}$ must be finite. We have that only finite subsets of $\{I_n : n \in A\}$ are bounded, therefore $\mathcal{S}$ contains an infinite sun set. This shows that for any web set $\mathcal{W} \subseteq \mathcal{I}$ there is some $n \in \omega$ such that $\mathcal{W} \subseteq \mathcal{K}_n$.

We now prove that $\mathcal{K}^{\downarrow}$ is a τ_{bd}-compact set if $\mathcal{K}$ is, which is enough to prove the second part. Indeed, let $\mathcal{K}$ be a τ_{bd}-compact set, then $\mathcal{K}^{\downarrow}$ is a web set since $\mathcal{K}$ is. Let $\{A_n : n \in \omega\} \subseteq \mathcal{K}^{\downarrow}$, then there is a family $\{B_n : n \in \omega\} \subseteq \mathcal{K}$ such that $A_n \subseteq B_n$ for all $n \in \omega$. Since 2^ω and $\mathcal{K}$ are compact sets, then there are a pair of subsequences $\{A_{n_k} : k \in \omega\}$, $\{B_{n_k} : k \in \omega\}$ which respectively converge to $X \in 2^\omega$ and $Y \in \mathcal{K}$. Finally, it is easy to see that $X \subseteq Y$, then $\mathcal{K}^{\downarrow}$ is a compact set and therefore it is a τ_{bd}-compact set. ■

Now we can give a combinatorial characterization for the σ-compactness of the bounded topology.

Theorem 5.2. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space if and only if $\mathcal{I}$ has the web-closure property and $\mathrm{web}(\mathcal{I}) = \omega$.

Proof. If $\mathcal{I}$ has the web-closure property and $\mathrm{web}(\mathcal{I}) = \omega$, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space by Proposition 4.6. On the other hand, if $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space then clearly $\mathrm{web}(\mathcal{I}) = \omega$. Let $\{\mathcal{K}_n : n \in \omega\}$ the family given by the previous lemma. We have that any web set of $\mathcal{I}$ is contained in $\mathcal{K}_n$ for some $n \in \omega$. Thus, again by Proposition 4.6, $\mathcal{I}$ has the web-closure property. ■

Using Corollary 4.13, we can write the previous result as follows.

Theorem 5.3. Let $\mathcal{I}$ be an ideal. Then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space if and only if $\mathcal{I}$ has the web-closure property and $\emptyset \times \mathrm{Fin} \not\leq_T \mathcal{I}$.

Using Lemma 5.1 we can improve the properties of the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ when it is σ-compact.

Theorem 5.4. Let $\mathcal{I}$ be an ideal. If the space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is σ-compact then it is a zero-dimensional topological group.

Proof. Let $\{\mathcal{K}_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{I})$ be the family given by Lemma 5.1.

For increasing $f \in \omega^{\leq \omega}$, let

$$\mathcal{U}_f = \{A \in \mathcal{I} : (\forall n \in \mathrm{dom}(f))\, A \cap f(n) \in \mathcal{K}_n\}.$$

Then $\mathcal{U}_f$ is a τ_{bd}-clopen set since $\mathcal{U}_f \cap \mathcal{P}(I)$ is clopen in $\mathcal{P}(I)$ for all $I \in \mathcal{I}$. For instance, fix $I \in \mathcal{I}$ and $J \in \mathcal{U}_f \cap \mathcal{P}(I)$; there is $m \in \omega$ such that $I \in \mathcal{K}_m$. Without loss of generality we may assume that $\mathrm{dom}(f) \geq m + 1$. Letting $s = \chi_{J \cap f(m+1)}$, it is not hard to see that $\langle s \rangle \cap \mathcal{P}(I) \subseteq \mathcal{U}_f$ for if $X \in \langle s \rangle \cap \mathcal{P}(I)$, then $X \cap f(k) = J \cap f(k) \in \mathcal{K}_j$, for all $k \leq m$ and $X \in \mathcal{K}_k$ for $k > m$ as $I \in \mathcal{K}_k = \mathcal{K}_k^{\downarrow}$.

We will show that $\{\mathcal{U}_f : f \in \omega^\omega \text{ is increasing}\}$ is a local base at $\emptyset$ in $(\mathcal{I}, \tau_{\mathrm{bd}})$. Let $\mathcal{U}$ be an τ_{bd}-open neighbourhood of $\emptyset$. We will recursively define sequence

$\{s_n : n \in \omega\}$ in order to get a map $f = \bigcup_{n\in\omega} s_n \in \omega^\omega$ such that $\mathcal{U}_f \subseteq \mathcal{U}$. Since $\mathcal{K}_0 \cap \mathcal{U}$ is open in $\mathcal{K}_0$ and $\emptyset$ belongs to it, there is some n_0 such that if $t \in \omega^{n_0}$ is the function with constant value zero, then $\langle t\rangle \cap \mathcal{K}_0 \subseteq \mathcal{U}$. Let $s_0(0) = n_0$. Suppose that $s_k \in \omega^{k+1}$ is already defined, say $s_k(i) = n_i$ for $i \leq k$, and $\mathcal{U}_{s_k} \cap \mathcal{K}_k \subseteq \mathcal{U}$. We claim that there is some $n_{k+1} > n_k$ such that

$$\mathcal{U}_{s_k^\frown n_{k+1}} \cap \mathcal{K}_{k+1} \subseteq \mathcal{U} \tag{$*$}$$

If there is no such n_{k+1}, then for all $n \in \omega \setminus (n_k + 1)$ there is some $I_n \in (\mathcal{U}_{s_k} \cap \mathcal{K}_{k+1}) \setminus \mathcal{U}$ such that $I_n \cap n \in \mathcal{K}_k$. Since $\mathcal{K}_{k+1}$ is a τ_{bd}-compact set, there is a subsequence $\left\{I_{n_j} : j \in \omega\right\} \subseteq \mathcal{I} \setminus \mathcal{U}$ which τ_{bd}-converges to some $I \in \mathcal{K}_{k+1}$. However, the sequence $\left\{I_{n_j} \cap n_j : j \in \omega\right\} \subseteq \mathcal{U}_{s_k} \cap \mathcal{K}_k$ also converges to I. That implies $I \in \mathcal{U}_{s_k} \cap \mathcal{K}_k$, since $\mathcal{U}_{s_k} \cap \mathcal{K}_k$ is τ_{bd}-closed; hence $I \in \mathcal{U}$, but this contradicts that $\mathcal{U}$ is an τ_{bd}-open set, thus ($*$) holds for some $n_{k+1} > n_k$ and we let $s_{k+1} = s_k{}^\frown\langle n_{k+1}\rangle$.

Hence, $\{\mathcal{U}_f : f \in \omega^\omega$ is increasing$\}$ will be a local base for neighbourhoods of $\emptyset$ formed by τ_{bd}-clopen sets. It follows that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a zero-dimensional space since it is homogeneous. Moreover, if $f \in \omega^\omega$ is increasing, letting $g(n) = f(2n)$, for all $n \in \omega$, one gets that $\mathcal{U}_g \triangle \mathcal{U}_g \subseteq \mathcal{U}_f$; from which it follows that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a zero-dimensional topological group. ∎

We will use the following lemmas to show that all σ-compact and no locally compact spaces $(\mathcal{I}, \tau_{\mathrm{bd}})$ are homeomorphic to $(\mathrm{Fin}\times\emptyset, \tau_{\mathrm{bd}})$ (Theorem 5.8), which is one of them since it is countably generated.

Lemma 5.5. Let $\mathcal{I} \neq \mathrm{Fin}$ be an ideal such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is σ-compact. Then there exist an increasing family of τ_{bd}-compact sets $\{\mathcal{K}_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{I})$ which covers $\mathcal{I}$ such that $\mathcal{K}_n = \mathcal{K}_n^{\downarrow}$ and $(\mathcal{K}_n, \tau\restriction_{K_n})$ is homeomorphic to 2^ω for all $n \in \omega$.

Proof. Let $\{\mathcal{F}_n : n \in \omega\}$ be the family given by Lemma 5.1. Since $\mathcal{I} \neq \mathrm{Fin}$ we can suppose that any $\mathcal{F}_n$ is uncountable.

Let $n \in \omega$. By Cantor-Bendixon Theorem, there exists a perfect subset and a countable open subset of $\mathcal{F}_n$, namely $\mathcal{K}_n$ and $\mathcal{C}_n$ respectively, such that $\mathcal{F}_n = \mathcal{K}_n \cup \mathcal{C}_n$. Since $\mathcal{F}_n = \mathcal{F}_n^{\downarrow}$, then $I \in \mathcal{F}_n$ is a condensation point of $\mathcal{F}_n$ if and only if there is some $J \in \mathcal{F}_n \cap [\omega]^\omega$ such that $I \subseteq J$. Therefore $\mathcal{K}_n = \mathcal{K}_n^{\downarrow}$ and $\mathcal{C}_n \subseteq [\omega]^{<\omega}$. Also, for all $n \in \omega$ we have that $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$ because $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. Finally, since for all $F \in [\omega]^{<\omega}$ there exist $X \in \mathcal{I} \cap [\omega]^\omega$ such that $F \subseteq X$, then $\{\mathcal{K}_n : n \in \omega\} \subseteq \mathcal{P}(\mathcal{I})$ cover $\mathcal{I}$ and it is the desired family. ■

Lemma 5.6. Let $\mathcal{I}$ be an ideal and $\mathcal{K} \subseteq \mathcal{I}$ be a τ_{bd}-compact set. If $\mathcal{K}^{\downarrow}$ is τ_{bd}-nowhere dense, there is a τ_{bd}-compact set $\mathcal{K}' \subseteq \mathcal{I}$ such that $\mathcal{K} \subseteq \mathcal{K}'$ and $\mathcal{K}$ is τ_{bd}-nowhere dense in $\mathcal{K}'$.

Proof. Without loss of generality, we can assume that $\mathcal{K} = \mathcal{K}^{\downarrow}$. Since $\mathcal{K}$ has empty interior and $(\mathcal{I}, \tau_{\mathrm{bd}})$ is sequential, there is a bounded sequence $\mathcal{X} = \{F_n : n \in \omega\} \subseteq \mathcal{I} \setminus \mathcal{K}$ which converges to $\emptyset$. Let

$$\mathcal{K}' = \mathcal{K} \cup \{I \cup F_n : I \in \mathcal{K},\ n \in \omega\} \subseteq \mathcal{I}$$

Note that for all $I \in \mathcal{K}$ and $n \in \omega$, $I \cup F_n \notin \mathcal{K}$ since $\mathcal{K} = \mathcal{K}^{\downarrow}$ and $F_n \notin \mathcal{K}$. We claim that $\mathcal{K}'$ is the desired set.

$\mathcal{K}'$ is a web set since $\mathcal{K}$ is and $\mathcal{X}$ is bounded. To see $\mathcal{K}'$ is compact let $\mathcal{Y} \subseteq \mathcal{K}'$ be a countable subset, then for all $J \in \mathcal{Y}$ there are $I_J \in \mathcal{K}$ and $F_{n_J} \in \mathcal{X}$ such that $J = I_J \cup F_{n_J}$, since $\mathcal{K}$ is compact we can assume that $\mathcal{Z} = \{I_J : J \in \mathcal{Y}\} \subseteq \mathcal{K}$ converges to some $I \in \mathcal{K}$. We claim that $\mathcal{Y}$ has a convergent subsequence. Indeed, if there is an infinite subset $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $(\forall\, I \in \mathcal{Y}')\ F_{n_J} = F_N$ for some $N \in \omega$, then $\mathcal{Y}'$ converges to $I \cup F_N$. Then we can assume that $F_{n_J} \neq F_{n_L}$ for any distinct $J, L \in \mathcal{Y}$. Let $n \in \omega$, since $\mathcal{Z}$ converges to I and $\mathcal{X}$ converges to $\emptyset$, we have that $(\forall^\infty\, J \in \mathcal{Y})\ I_J \in \langle I {\restriction}_n \rangle$ and $(\forall^\infty\, J \in \mathcal{Y})\ F_{n_J} \cap n = \emptyset$. Then $(\forall\, n \in \omega)\,(\forall^\infty\, J \in \mathcal{Y})\ J \in \langle I {\restriction}_n \rangle$, hence $\mathcal{Y}$ converges to I. Therefore $\mathcal{K}' \subseteq \mathcal{I}$ is τ_{bd}-compact.

Finally, let $\mathcal{U} \subseteq \mathcal{K}'$ be an τ_{bd}-open set of $\mathcal{K}'$ such that there is some $I \in \mathcal{U} \cap \mathcal{K}$. Since $\{I \cup F_n : n \in \omega\} \subseteq \mathcal{K}'$ τ_{bd}-converges to I, there is some $J \in \mathcal{U} \setminus \mathcal{K}$. Since $\mathcal{K}$ is τ_{bd}-closed, there is some τ_{bd}-open set $\mathcal{V}$ such that $J \in \mathcal{V}$ and $\mathcal{V} \cap \mathcal{K} = \emptyset$. Hence, $\mathcal{V} \cap \mathcal{U} \subseteq \mathcal{U}$ is a non-empty τ_{bd}-open set of $\mathcal{K}'$ disjoint to $\mathcal{K}$. Therefore $\mathcal{K}$ is τ_{bd}-nowhere dense in $\mathcal{K}'$. ■

We also need the following result due to B. Knaster and M. Reichbach[4]

Theorem 5.7 (Knaster and Reichbach, [16, Theorem 2])**.** Let X, Y be a pair of compact, perfect, zero-dimensional and metric spaces, and let $X' \subseteq X$, $Y' \subseteq Y$ be closed nowhere dense subsets of its respective space. If $\varphi' : X' \to Y'$ is a homeomorphism, then there exists a homeomorphism $\varphi : X \to Y$ extending φ'.

Theorem 5.8. Let $\mathcal{I}$ be an ideal such that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is a σ-compact space. Then either $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{P}(A)$ for some $A \in \mathcal{I}$ or $(\mathcal{I}, \tau_{\mathrm{bd}})$ is homeomorphic to $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$.

Proof. Suppose that $\mathcal{I} \neq \mathrm{Fin} \oplus \mathcal{P}(A)$ for any $A \in \mathcal{I}$. Let $\{\mathcal{K}_n : n \in \omega\}$ the family given by Lemma 5.5, since $\mathcal{I}$ is no locally compact then every $\mathcal{K}_n$ is τ_{bd}-nowhere dense. Using Lemma 5.1 and Lemma 5.6 we can assume that $\mathcal{K}_n$ is a τ_{bd}-nowhere dense subspace of $\mathcal{K}_{n+1}$ for all $n \in \omega$. Let $C_n = (n+1) \times \omega \in \mathrm{Fin} \times \emptyset$. Using the previous theorem, we have that for all $n \in \omega$ there is a τ_{bd}-homeomorphism $\varphi_n : \mathcal{K}_n \to \mathcal{P}(C_n)$ such that φ_{n+1} extends φ_n. Let $\varphi = \bigcup_{n\in\omega} \varphi_n$, we claim that the bijective map $\varphi : \mathcal{I} \to \mathrm{Fin} \times \emptyset$ is a τ_{bd}-homeomorphism.

Let $\mathcal{U} \subseteq \mathrm{Fin} \times \emptyset$ be an τ_{bd}-open set. Since φ is injective,

$$\varphi^{-1}[\mathcal{U}] \cap \mathcal{K}_n = \varphi_n^{-1}[\mathcal{U} \cap \mathcal{C}_n]$$

and then $\varphi^{-1}[\mathcal{U}] \cap \mathcal{K}_n$ is open in $\mathcal{K}_n$ for all $n \in \omega$. Therefore $\varphi^{-1}[\mathcal{U}] \subseteq \mathcal{I}$ is a τ_{bd}-open set because the family $\{\mathcal{K}_n : n \in \omega\}$ is cofinal among the web sets of $\mathcal{I}$ by Lemma 5.1. Thus φ is a τ_{bd}-continuous map. Analogous arguments for φ^{-1} shows that φ is an τ_{bd}-open map, therefore it is a τ_{bd}-homeomorphism. ■

Note that the homeomorphism given in the previous result does not preserve cofinal subsets, because $\mathrm{cof}(\mathrm{Fin} \times \emptyset) = \omega < \mathrm{cof}(\mathcal{J}_{\mathcal{P}})$ although $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ and $(\mathcal{I}_{\mathcal{P}}, \tau_{\mathrm{bd}})$ are homeomorphic spaces. We have the following question.

Question 5.9. Theorem 5.8 inplies that all spaces $(\mathcal{I}, \tau_{\mathrm{bd}})$ which are σ-compact are homeomorohic. Are they equivalents as topological groups?

[4] Originally due to C. Ryll-Nardzewski answering a problem by B. Knaster.

Summarizing, the space $(\mathrm{Fin}\times\emptyset,\tau_{\mathrm{bd}})$ is homogeneous, separable, sequential, σ-compact, non-compact, zero-dimensional, topological group, and not Fréchet–Urysohn and hence neither metrizable nor second-countable.

The Polish space known as the *complete Erdös space* $\mathfrak{E}_c$, defined as the closed subspace of ℓ^2 such that $(x_n)_{n\in\omega}\in\mathfrak{E}_c$ if and only if $(\forall\, n\in\omega)\ x_n\in\left\{{}^1\!/_m : m\in\omega\right\}\cup\{0\}$, was introduced by P. Erdös in [6]. It is a totally disconnected, one-dimensional and almost zero dimensional space (see [4]). J. Dijkstra and J. van Mill proved that if $\mathcal{I}$ is an F_σ P-ideal, then $(\mathcal{I},\tau_{\mathrm{bd}})$ is not a σ-compact space if and only if it is homeomorphic to $\mathfrak{E}_c$ (see [3, Theorem 4.15]). Using this result we prove the following.

Theorem 5.10. Let $\mathcal{I}$ be an F_σ-ideal. Then the space $(\mathcal{I},\tau_{\mathrm{bd}})$ is homeomorphic to $\mathfrak{E}_c$ if and only if $\mathcal{I}$ is a P-ideal and $\emptyset\times\mathrm{Fin}\leq_T\mathcal{I}$.

Proof. If $\mathcal{I}$ is a P-ideal such that $\emptyset\times\mathrm{Fin}\leq_T\mathcal{I}$, then $(\mathcal{I},\tau_{\mathrm{bd}})$ is no σ-generated by Theorem 5.3, and hence $(\mathcal{I},\tau_{\mathrm{bd}})$ is homeomorphic to $\mathfrak{E}_c$. On the other hand, if $(\mathcal{I},\tau_{\mathrm{bd}})$ is homeomorphic to $\mathfrak{E}_c$ then the space is metrizable, hence $\mathcal{I}$ is a P-ideal by Theorem 2.5. By Proposition 3.4 the ideal $\mathcal{I}$ has the web-closure property. Then, again by Theorem 5.3, we conclude that $\emptyset\times\mathrm{Fin}\leq_T\mathcal{I}$ since $(\mathcal{I},\tau_{\mathrm{bd}})$ is no a σ-compact space. ■

Let $\mathcal{I}$ be an F_σ and P-ideal. Using a theorem by K. Mazur in [21], there is a lower semicontinuous submeasure φ such that $\mathcal{I}=\mathrm{Fin}\,(\varphi)=\{A\subseteq\omega:\varphi(A)<0\}$. For $\varepsilon>0$, if $\{A\subseteq\omega:\varphi(A)=\varepsilon\}=\emptyset$ then $\{A\subseteq\omega:\varphi(A)<\varepsilon\}$ is τ_{bd}-clopen. So, if $\emptyset\times\mathrm{Fin}\leq_T\mathcal{I}$, there exists $\varepsilon>0$ such that $(\forall\, x\in[0,\varepsilon])\,(\exists\, A\subseteq\omega)\ \varphi(A)=x$, because otherwise $(\mathcal{I},\tau_{\mathrm{bd}})\simeq\mathfrak{E}_c$ would be a zero-dimensional space.

Using some of the previous results, we give a classification of F_σ-ideals through its bounded topologies as follows.

Theorem 5.11. Let $\mathcal{I}$ be an F_σ-ideal. Then:

i) $(\mathcal{I},\tau_{\mathrm{bd}})\simeq\omega$ if and only if $\mathcal{I}=\mathrm{Fin}$.

ii) $(\mathcal{I},\tau_{\mathrm{bd}})\simeq\omega\times 2^\omega$ if and only if $\mathcal{I}=\mathrm{Fin}\oplus\mathcal{P}\,(A)$ for some infinite $A\in\mathcal{I}$.

iii) $(\mathcal{I}, \tau_{\text{bd}}) \simeq (\text{Fin} \times \emptyset, \tau_{\text{bd}})$ if and only if $\mathcal{I}$ has the web-closure property, $\emptyset \times \text{Fin} \not\leq_T \mathcal{I}$ and $\mathcal{I} \neq \text{Fin} \oplus \mathcal{P}(A)$ for any $A \in \mathcal{I}$.

iv) $(\mathcal{I}, \tau_{\text{bd}}) \simeq \mathfrak{E}_c$ if and only if $\mathcal{I}$ is a P-ideal and $\emptyset \times \text{Fin} \leq_T \mathcal{I}$.

We conclude with some conjectures related with the previous result.

Conjecture 5.12. Let $\mathcal{I}$ be an F_σ-ideal whose bounded topology does not satisfy any of the conditions in Theorem 5.11. Must $\mathcal{I}$ have a strong unbounded set of size $\mathfrak{c}$?

If the previous is true then that would imply Conjecture 4.14 also in the positive.

Question 5.13. Is there a result analogous to Theorem 5.11 for P-ideals?

About the previus question, we conjecture the following.

Conjecture 5.14. Let $\mathcal{I}$ be a P-ideal such that $(\mathcal{I}, \tau_{\text{bd}})$ is no σ-compact. Then $(\mathcal{I}, \tau_{\text{bd}})$ is homeomorphic either to ω^ω, $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$.

6. A test space

In his recent visit to Morelia, Alexander Shibakov pointed to us the importance of the following space to the structure of sequential topological groups (see [11], [12]). We are really thankful to him since his insight opened up some new paths.

- **Convergent sequence of discrete sets** (see [30]). Denoted by $D(\omega)$ is the set $\omega \times \omega \cup \{(\omega, \omega)\}$ endowed with the topology that makes $\omega \times \omega$ discrete and such that the set $\{(\omega \setminus k) \times \omega \cup (\omega, \omega) : k \in \omega\}$ is a local basis for (ω, ω).

The following related result is due to T. Banakh and L. Zdomskyĭ.

Theorem 6.1 (Banakh and Zdomskyĭ, [1]). Let $\mathbb{G}$ be a sequential group in which every point is G_δ. If $\mathbb{G}$ contains a closed copy of $D(\omega)$ then it is Fréchet.

Since any space $(\mathcal{I}, \tau_{\mathrm{bd}})$ is sequential, every point $I \in \mathcal{I}$ is τ_{bd}-closed and by Theorem 2.5; we have the following.

Theorem 6.2. Let $\mathcal{I}$ be a non P-ideal. If $(\mathcal{I}, \tau_{\mathrm{bd}})$ contains a closed copy of $D(\omega)$, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ is not a topological group.

An ideal $\mathcal{I}$ is an P^+-ideal if for every decreasing sequence $\{A_n : n \in \omega\} \subseteq \mathcal{I}^+$ there is $A \in \mathcal{I}^+$ such that $A \subseteq^* A_n$ for all $n \in \omega$. We have the following about these ideals.

Proposition 6.3. Let $\mathcal{I}$ be a non P^+-ideal, then $(\mathcal{I}, \tau_{\mathrm{bd}})$ contains a closed copy of $D(\omega)$.

Proof. Let $\mathcal{A} = \{A_n : n \in \omega\} \subseteq \mathcal{P}(\omega) \setminus \mathcal{I}$ be a decreasing family witness that $\mathcal{I}$ is no an P^+-ideal, we can assume that $A_n \setminus A_{n+1} \notin \mathcal{I}$ for all $n \in \omega$. As before, let $A(k)$ be the k-th element of a set $A \subseteq \omega$ then for all $n, m \in \omega$ let $I_n^m = \{(A_n \setminus A_{n+1})(k) : k \leq m\} \in \mathcal{I}$. We claim that $\mathcal{D} = \{I_n^m : n, m \in \omega\} \cup \{\emptyset\}$ is a closed copy of $D(\omega)$ in $(\mathcal{I}, \tau_{\mathrm{bd}})$.

For a fixed $n \in \omega$, the set $\mathcal{S}_n = \{I_n^m : m \in \omega\} \subseteq \mathcal{I}$ is a sun set since any infinite subset is unbounded, therefore $\mathcal{S}_n$ is a discrete set in the space $(\mathcal{I}, \tau_{\mathrm{bd}})$. Now, for any map $f : \omega \to \omega$, the set $\{I_n^{f(n)} : n \in \omega\}$ is disjoint and bounded, since is a pseudo-intersection of $\mathcal{A}$, therefore it τ_{bd}-converges to $\emptyset$. This shows that $\mathcal{D}$ satisfies what is desired. ■

It is easy to find a copy of $D(\omega)$ inside $(\mathrm{Fin} \times \mathrm{Fin}, \tau_{\mathrm{bd}})$. Indeed, fix an infinite partition of ω formed by infinite subsets, say $\{A_m : n \in \omega\}$, and set $d(m, n) = A_m \times \{n\}$, for all $m, n \in \omega$. Then $\{d(m, n) : m, n \in \omega\} \cup \{\emptyset\}$ is a copy of $D(\omega)$. The ideal $\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists m, n \in \omega)\,(\forall k > m)\,|A \cap (\{k\} \times \omega)| \leq n\}$ also has a copy of $D(\omega)$. As in the case of $\mathrm{Fin} \times \mathrm{Fin}$, considering an infinite partition of ω into infinite sets and using the same sets $d(m, n)$ we have the columns of $D(\omega)$. In this case, any transversal selection has size one on each column, thus its union is an element of the ideal, then it τ_{bd}-converging to $\emptyset$. Moreover, the ideal $\mathcal{ED}$ has a sun set of size $\mathfrak{c}$ ([22, Theorem 1.6.4]).

The following is known as the *branching ideal*

$$\mathcal{B}r = \langle \{ \{x\restriction_n : n \in \omega\} \subseteq 2^{<\omega} : x \in 2^\omega \} \rangle.$$

The space $(\mathcal{B}r, \tau_{\text{bd}})$ does not has a copy of $D(\omega)$. To see this, note that for any countable family $\{A_n \subseteq 2^{<\omega} : A_n \text{ is antichain}\}$ we can recursively define a function $x \in 2^\omega$ such that for all $k \in \omega$ there are infinitely many $n \in \omega$ such that $|A_n \cap \langle x\restriction_k \rangle| = \omega$. Therefore, if $D(\omega)$ is embedding in $(\mathcal{B}r, \tau_{\text{bd}})$ then any column of $D(\omega)$ contains an antichain, and by previous we can find a transversal selection which is sun set, therefore it does not τ_{bd}-converge. Moreover, $\{\{x\restriction_n : n \in \omega\} : x \in 2^\omega\} \subseteq \mathcal{B}r$ is a sun set of size $\mathfrak{c}$.

As mentioned in [20, Proposition 5.9], A. Louveau and B. Veličković noted in [19] that any ideal $\mathcal{I}$ with a sun set of size $\mathfrak{c}$ is Tukey-top (that is, $\mathcal{J} \leq_T \mathcal{I}$ for any ideal $\mathcal{J}$, in particular any two Tukey-top ideals are Tukey-equivalent). Then the ideals $\mathcal{B}r$ and $\mathcal{ED}$ are witnesses for the following.

Proposition 6.4. There exists a pair of Tukey-equivalent ideals such that its bounded topologies are not homeomorphic.

We know that the regularity of the bounded topology does not imply that the space is a topological group. As an example, $(\mathcal{I}_{1/n} \oplus \text{Fin} \times \emptyset, \tau_{\text{bd}})$, where $\oplus$ denotes the disjoint sum of ideals, is regular because $(\mathcal{I}_{1/n}, \tau_{\text{bd}})$ and $(\text{Fin} \times \emptyset, \tau_{\text{bd}})$ are. Also, it is not a topological group because $(\mathcal{I}_{1/n}, \tau_{\text{bd}})$ has a closed copy of $S(\omega)$ and $(\text{Fin} \times \emptyset, \tau_{\text{bd}})$ has a closed copy of $D(\omega)$, therefore $(\mathcal{I}_{1/n} \oplus \text{Fin} \times \emptyset, \tau_{\text{bd}})$ has a closed copy of both spaces and, by Theorem 6.2, it is not a topological group.

Proposition 6.5. Let $\mathcal{I}$ be a maximal ideal, then $(\mathcal{I}, \tau_{\text{bd}})$ is a topological group if and only if $\mathcal{I}$ is a P-ideal.

Proof. Any maximal ideal $\mathcal{I}$ is non-meager. So, if it is a P-ideal then its bounded topology is the topology induced by 2^ω (Theorem 2.6). On the other hand, by maximality, if $\mathcal{I}$ is a non P-ideal then it is a non P^+-ideal. So, $(\mathcal{I}, \tau_{\text{bd}})$ contains closed copies of $S(\omega)$ and $D(\omega)$. Therefore it is not a topological group. ∎

7. Open problems

There seem to be many interesting directions for further research about the bounded topology, several of them directly related to the results of the paper. The first group of problems asks about combinatorial translations/charactarizations of natural topological properties of ideals endowed with the bounded topology.

Question 7.1. For which (Borel) ideals $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ a topological group?

One has to wonder if for Borel ideals this happens if and only if the ideals have the web-closure property, in particular if such ideals have to be either P-ideals or σ-weakly bounded ones.

A related question is:

Question 7.2. For which (Borel) ideals $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ regular?

For the following question one would suspect that $(\mathcal{I}, \tau_{\mathrm{bd}})$ is Lindelöf if and only if every strongly unbounded subset of $\mathcal{I}$ is countable.

Question 7.3. For which (Borel) ideals $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ Lindelöf?

The relationship between separability and the Lindelöf property is one of István Juhász's favourite subjects (see e.g. [8], [14]).

An interesting question is to give an external characterization of spaces of the type $(\mathcal{I}, \tau_{\mathrm{bd}})$. We know they have to be homogeneous, separable and sequential with a weaker homogeneous zero-dimensional metric topology.

Question 7.4. Which topological spaces are homeomorphic to $(\mathcal{I}, \tau_{\mathrm{bd}})$ for some (Borel) ideal $\mathcal{I}$?

We know they have to be homogeneous, separable and sequential with a weaker homogeneous zero-dimensional metric topology. Is this sufficient?

A related problem also asks about the variety of these examples:

Question 7.5. Are there infinitely (uncountably) many Borel ideals $\mathcal{I}$ such that the spaces $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ are mutually non-homeomorphic?

Another series of problems deals with non-definable ideals

Question 7.6. For which ideals $\mathcal{I}$ is $(\mathcal{I}, \tau_{\mathrm{bd}})$ metrizable?

We know that such ideals would have to be P-ideals and we know that for analytic and non-meager ones this characterizes metrizability but in general we do not know. So, in particular, we do not know if there is (even consistently) an ideal which is Fréchet-Urysohn but not metrizable.

Question 7.7. Is it consistent that all ultrafilters (or rather all maximal ideals) when endowed with the bounded topology are mutually homeomorphic?

Finally, we repeat the probably most interesting questions mentioned already in the text which ask about complete topological classification of ideals which are F_σ, resp. P-ideals:

Question 7.8. Is every analytic P-ideal with the bounded topology homeomorphic to one of ω, $\omega \times 2^\omega$, ω^ω, $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$?

Question 7.9. Is every F_σ-ideal without a perfect strongly unbounded subset with the bounded topology homeomorphic to one of ω, $\omega \times 2^\omega$, $(\mathrm{Fin} \times \emptyset, \tau_{\mathrm{bd}})$ or $\mathfrak{E}_c$?

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