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Realcompactness in maximal and submaximal spaces

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Abstract

We study realcompactness in the classes of submaximal and maximal spaces. It is shown that a normal submaximal space of cardinality less than the first measurable is realcompact. ZFC examples of submaximal not realcompact and maximal not realcompact spaces are constructed. These examples answer questions posed in [O.T. Alas, M. Sanchis, M.G. Tkačenko, V.V. Tkachuk, R.G. Wilson, Irresolvable and submaximal spaces: homogeneity versus σ-discreteness and new ZFC examples, Topology Appl. 107 (3) (2000) 259–273] and generalize some results from [D.P. Baturov, On perfectly normal dense subspaces of products, Topology Appl. 154 (2) (2007) 374–383].

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0. Introduction

The class of submaximal spaces was introduced by N. Bourbaki in Topologie Générale. We recall that a subset A of a space X is *locally closed* if A is open in its closure in X or, equivalently, is the intersection of an open subset and a closed subset of X.

Definition 1. A space X is a submaximal space if every subset of X is locally closed.

Independently, Hewitt [8] defined a space to be an MI-space if it has no isolated points and every dense subset is open. Hewitt showed that every subset of a space is locally closed if and only if every dense subset is open, hence we adopt the convention that a space is submaximal if it has no isolated points and every dense subset is open. In the

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same paper, Hewitt defined a space to be *maximal* if its topology is maximal in the collection of all topologies on X with no isolated points. The existence of a maximal space that is Tychonoff is nontrivial and due to van Douwen [6].

The paper of Arhangel'skiĭ and Collins [2] gave the first comprehensive study of the class of submaximal spaces. There a number of questions were raised and answered in a subsequent paper of Alas et al. [1]. In the second paper it was asked whether (ZFC) submaximal dense subsets of I^{2^κ} exist for cardinals $\kappa \geqslant \aleph_0$ (interestingly, there can be no such maximal spaces). Also, it was asked whether maximal or submaximal spaces are realcompact. In this note we provide a number of ZFC examples answering these questions. In addition it is shown that normal submaximal spaces of cardinality less than the first measurable cardinal are realcompact.

1. Normal submaximal spaces

In this section we prove that if X is normal, submaximal and of cardinality less than the first measurable cardinal, then X is realcompact. Thus any ZFC example of a submaximal realcompact space cannot be provably normal.

Recall that a point a of $\beta X \setminus X$ is called a *far point* if a is not in the closure of any discrete closed subset of X.

Theorem 2. *A normal submaximal space of a non-Ulam-measurable cardinality is realcompact.*

Proof. Let X be as in the wording of the theorem and let $a \in \beta X \setminus X$. According to [7], it is enough to prove that there is a G_δ set in βX that contains a and is disjoint from X. Denote the family of all the open neighborhoods of a in βX by Φ.

Case I. a is not a far point. Let $D \subset X$ be discrete and closed subset of X such that $a \in \overline{D}$. Because X is normal, the set $\Phi_D = \{F \cap D\colon F \in \Phi\}$ is an ultrafilter on D. Φ_D is not countably centered because $|D|$ is not Ulam-measurable. Let $\{D_n \in \Phi_D\colon n \in \omega\}$ be a countable family of subsets that has empty intersection. For every $n \in \omega$, the sets $D_{n+1} \setminus D_n$ and D_n are functionally disjoint, therefore there are $U_n \in \Phi$ such that $D_n \subset U_n$ and $(D_{n+1} \setminus D_n) \cap \overline{U_n} = \emptyset$. The set $\bigcap_{n\in\omega} \overline{U_n}^X$, which we denote by H, is a closed subset of X that is disjoint from D. Now, $a \notin \overline{H}^{\beta X}$ because $a \in \overline{D}$ and because X is normal. Hence, the set $U_{-1} = \beta X \setminus \overline{H}^{\beta X}$ is an element of Φ. This implies that $\{U_n\colon n \geqslant -1\}$ is a family of open neighborhoods of a in βX whose intersection is disjoint from X as required.

Case II. a is a far point. By 5.2 from [5], a is not in the closure of any two disjoint open sets, hence the family $\Psi = \{U \text{ is open in } X\colon a \in \overline{U}\}$ is an ultrafilter of open subsets of X. Hence, Ψ contains a base $\mathcal{B}$ for a set-theoretic ultrafilter (Corollary 20 from [11]). This ultrafilter is not countably centered because the cardinality of X is not Ulam-measurable. Therefore, there is a countable family $\{V_n \in \mathcal{B}\colon n \in \omega\}$ such that $(\bigcap_{n\in\omega} V_n) \cap X = \emptyset$. The proof will be complete when we show that for every $n \in \omega$ there is $W_n \in \Phi$ such that $W_n \cap X \subset V_n$. Indeed, let $V_n' = \operatorname{Int}^{\beta X}(\overline{V_n}^{\beta X})$. The set $H_n = (V_n' \setminus V_n) \cap X$ is a nowhere dense, hence discrete subset of X. So it does not contain a in its closure since a is a far point. Again by 5.2 from [5], the set $X \setminus V_n'$ does not contain a in its closure. Therefore, the set $W_n = \beta X \setminus (\overline{H_n}^{\beta X} \cup \overline{X \setminus V_n'}^{\beta X}) = \beta X \setminus \overline{X \setminus V_n}^{\beta X}$ is an open neighborhood of a in βX. Clearly, $W_n \cap X = V_n$. $\square$

2. Dense submaximal subspaces of I^{2^κ}

Recently Juhasz, Soukup and Szentmiklóssy [9] proved that for any κ the Cantor cube 2^{2^κ} includes a dense submaximal space of cardinality κ. From this they concluded that $[0,1]^{\mathfrak{c}}$ includes a countable dense submaximal space. Here we give a direct construction of a dense submaximal space in M^{2^κ} for every separable metric space M consisting of more than one point and for every κ. The construction is flexible enough to also directly give dense submaximal not realcompact subspaces of I^{2^κ} thus answering two questions from [1].

Theorem 3. *Let κ be an infinite cardinal and let M be a separable metric space consisting of more than one point. If Z is a subset of M^{2^κ} of cardinality κ, then there is a normal dense submaximal subspace Y of $M^{2^\tau} \setminus Z$ such that every discrete subspace of Y is closed in $Z \cup Y$.*

Proof. It is known that a dense-in-itself space X is submaximal if and only if it is open-hereditarily dense (that is, no nonempty open subset of X can be represented as a union of two disjoint dense subsets) and nodec (every nowhere dense subset of X is discrete and closed in X). Hence, the desired set Y is submaximal whenever it satisfies both of the following conditions:

(i) For every disjoint subsets A_0, A_1 of Y, the set $\overline{A_1} \cap \overline{A_2}$ (both closures are taken in M^{2^κ}) is nowhere dense in M^{2^κ}.
(ii′) For every $a \in Y$ and every nowhere dense subset B of $Y \setminus \{a\}$, $a \notin \overline{B}$.

Both (ii′) and the condition that every discrete subspace of Y is closed in $Z \cup Y$ follow from the following condition (ii):

(ii) For every $a \in Z \cup Y$ and every nowhere dense subset B of $Y \setminus \{a\}$, $a \notin \overline{B}$.

Malykhin proved that Y is normal iff it satisfies condition (iii):

(iii) For every two disjoint discrete subsets A_0, A_1 of Y, there is a continuous function $f : Y \to [0,1]$ such that $f|_{A_0} \equiv 0$ and $f|_{A_1} \equiv 1$.

We will construct dense $Y \subset M^{2^\kappa}$ that satisfies (i)–(iii). Start with a dense $X_0 \subset M^{2^\kappa} \setminus Z$ of size κ such that every point of X_0 differs from every other point of X_0 and from every point of Z on 2^κ coordinates. Define sets $X_\alpha \subset M^{2^\kappa} \setminus Z$ for every α with $1 \leqslant \alpha \leqslant 2^\kappa$. If α is a successor ordinal, X_α will be obtained from X_{α^-} by changing the αth coordinate of some points of X_{α^-}. If α is a limit ordinal, X_α will be the limit of X_β, $\beta < \alpha$ as described below. The desired set X_{2^κ} will also be denoted by Y. To assure that Y remains disjoint from Z, let us assume that the indices which witness $Y \cap Z = \emptyset$ are all limit ordinals of 2^κ and we will only modify the points of X on successor ordinals. Also, we can assume that M contains at least two nonisolated points (if not, replace M with M^ω), which we denote by p_0 and p_1. For every $x \in M^{2^\kappa}$, the γth coordinate is denoted by $\pi_\gamma(x)$ and $x(\gamma)$. Let

$$\{(A^0_\alpha, A^1_\alpha)\colon \alpha \text{ is an even successor and } \alpha < 2^\kappa\}$$

be an enumeration of all disjoint pairs of infinite disjoint subsets of κ. Also, let

$$\{(B_\alpha, a_\alpha)\colon \alpha \text{ is an odd successor and } \alpha < 2^\kappa\}$$

be an enumeration of all pairs (B, a) such that B is a subset of κ and $a \in \kappa \cup Z$, $a \notin B$. We assume that in both enumerations each pair occurs 2^κ times.

Fix an enumeration of $X_0 = \{x^0_\eta\colon \eta < \kappa\}$. Let $\gamma \leqslant 2^\kappa$ and suppose that X_β have been defined for all $\beta < \gamma$ and enumerated by $X_\beta = \{x^\beta_\eta\colon \eta < \kappa\}$ subject to the following conditions

(1_β) If β is a successor ordinal, then $X_{\beta^-} \setminus X_\beta$ is a co-dense subset of M^{2^κ}.
(2_β) For every $\delta < \beta \leqslant 2^\kappa$ and every $\eta < \kappa$, if $x^\delta_\eta(\alpha) \neq x^\beta_\eta(\alpha)$, then $\delta < \alpha \leqslant \beta$.

Note that property (2_β) implies the following property (3_β)

(3_β) $\{\alpha \in 2^\kappa\colon x^{\beta^-}_\eta(\alpha) \neq x^\beta_\eta(\alpha)\} \subseteq \{\beta\}$ whenever β is a successor ordinal.

Case I. γ is an even successor ordinal. Denote the sets $\{x^{\gamma^-}_\eta\colon \eta \in A^0_\gamma\} \subset X_{\gamma^-}$ and $\{x^{\gamma^-}_\eta\colon \eta \in A^1_\gamma\} \subset X_{\gamma^-}$ by A_0 and A_1, respectively.

Subcase 1. The intersection $\overline{A_0} \cap \overline{A_1}$ (both closures are taken in M^{2^κ}) contains a nonempty open subset of M^{2^κ}, which we denote by A.

If $x^{\gamma^-}_\eta \in A_0 \cap A$ and $\alpha < 2^\kappa$, then

$$x^\gamma_\eta(\alpha) = \begin{cases} x^{\gamma^-}_\eta(\alpha) & \text{if } \alpha \neq \gamma, \\ p_0 & \text{if } \alpha = \gamma. \end{cases}$$

If $x^{\gamma^-}_\eta \in X_{\gamma^-} \setminus (A_0 \cap A)$, then $x^\gamma_\eta = x^{\gamma^-}_\eta$.

Note that $X_{\gamma^-} \setminus X_\gamma$ is a subset of $A_1 \cap A$, which is co-dense in M^{2^κ}, hence property (1_γ) holds. Also, $\pi_\gamma(\{x^\gamma_\eta\colon \eta \in A^0_\gamma\}) = \{p_0\}$, therefore $\{x^\gamma_\eta \restriction \gamma\colon \eta \in A^0_\gamma\}$ is a nowhere dense subset of $M^{\gamma+1}$.

Subcase 2. The sets A_0 and A_1 are disjoint and nowhere dense in M^{2^κ}. Then

$$x^\gamma_\eta(\alpha) = \begin{cases} x^{\gamma^-}_\eta(\alpha) & \text{if } \alpha \neq \gamma, \\ p_i & \text{if } \alpha = \gamma \end{cases}$$

whenever $\eta \in A^i_\gamma$ and $x^\gamma_\eta = x^{\gamma^-}_\eta$ if $\eta \notin A^0_\gamma \cup A^1_\gamma$.

In this subcase, $X_{\gamma^-} \setminus X_\gamma$ is a subset of $A_0 \cup A_1$, which is a nowhere dense subset of M^{2^κ}, hence property (1_γ) holds. Also, continuous function π_γ assumes constant values p_0 and p_1 on the sets $\{x^\gamma_\eta\colon \eta \in A^0_\gamma\}$ and $\{x^\gamma_\eta\colon \eta \in A^1_\gamma\}$, respectively. Therefore, there is a continuous function $f : X_\gamma \restriction \gamma \to [0,1]$ that assumes constant values 0 and 1 on sets $\{x^\gamma_\eta \restriction \gamma\colon \eta \in A^0_\gamma\}$ and $\{x^\gamma_\eta \restriction \gamma\colon \eta \in A^1_\gamma\}$, respectively.

Subcase 3. Subcases 1 and 2 fail. Put $x^\gamma_\eta = x^{\gamma^-}_\eta$ for every $\eta < \kappa$.

Property (3_γ) follows directly from the construction in each subcase. Together with (2_β), which is true for every $\beta < \gamma$, property (3_γ) implies (2_γ).

Case II. γ is an odd successor ordinal. Denote the set $\{x^{\gamma^-}_\eta\colon \eta \in B_\gamma\} \subset X_{\gamma^-}$ by B.

Subcase 1. $a_\gamma \in \kappa$, hence $x^{\gamma^-}_{a_\gamma} \in X_{\gamma^-}$. Denote either of the points p_0, p_1 that is distinct from $\pi_\gamma(x^{\gamma^-}_{a_\gamma})$ by p.

Let $x^\gamma_{a_\gamma} = x^{\gamma^-}_{a_\gamma}$. If B is nowhere dense in M^{2^κ} and $x^{\gamma^-}_{a_\gamma} \in \overline{B}$, then for every $\eta \in B_\gamma \setminus \{a_\gamma\}$,

$$x^\gamma_\eta(\alpha) = \begin{cases} x^{\gamma^-}_\eta(\alpha) & \text{if } \alpha \neq \gamma, \\ p & \text{if } \alpha = \gamma. \end{cases}$$

(Note that $X_{\gamma^-} \setminus X_\gamma$ is a subset of B, which is a nowhere dense subset of M^{2^κ}, hence property (1_γ) holds. Also, $x^\gamma_{a_\gamma} \restriction \gamma \notin \overline{(\{x^\gamma_\eta\colon \eta \in B_\gamma\}) \restriction \gamma}$.)

Otherwise, $x^\gamma_\eta = x^{\gamma^-}_\eta$ for every $\eta < \kappa$.

Subcase 2. $a_\gamma \in Z$. Denote either of points p_0, p_1 that is distinct from $\pi_\gamma(a_\gamma)$ by p.

If B is a nowhere dense subset of M^{2^κ} and $a_\gamma \in \overline{B}$, then for every $\eta \in B_\gamma$ and $\alpha < 2^\kappa$,

$$x^\gamma_\eta(\alpha) = \begin{cases} x^{\gamma^-}_\eta(\alpha) & \text{if } \alpha \neq \gamma, \\ p & \text{if } \alpha = \gamma. \end{cases}$$

(Note that $X_{\gamma^-} \setminus X_\gamma$ is a subset of B, which is a nowhere dense subset of M^{2^κ}, hence property (1_γ) holds. Also, $a_\gamma{}^\gamma \restriction \gamma \notin \overline{(\{x^\gamma_\eta\colon \eta \in B_\gamma\}) \restriction \gamma}$.)

Otherwise, $x^\gamma_\eta = x^{\gamma^-}_\eta$ for every $\eta < \kappa$.

The proof of (2_γ) in case II is the same as in case I.

Case III. γ is a limit ordinal. It follows from (3_β), $\beta < \gamma$, that for every $\eta < \kappa$ there is a unique point in M^{2^κ} that is a complete accumulation point of $\{x^\alpha_\eta\colon \alpha < \gamma\}$. Denote this point by x^γ_η.

Property (2_γ) is true because it is true for every $\beta < \gamma$.

Now we prove the desired properties of Y.

Assume towards contradiction that Y is not dense in M^{2^κ}. Let $\gamma \leqslant 2^\kappa$ be the smallest ordinal such that X_γ is not dense in M^{2^κ}. Then (1_γ) implies that γ is a limit ordinal. Pick a nonempty canonical neighborhood U of M^{2^κ} that misses X_γ. This neighborhood depends on a finite set $s \subset 2^\kappa$, so there is ordinal $\alpha < 2^\kappa$ such that $s \subset \alpha$. Due to minimality of γ, X_α is dense in M^{2^κ}, so there exist $x \in X_\alpha \cap U$. But then $x \in Y$ by (2_{2^κ}), which is a contradiction.

Prove (i). Assume towards contradiction that A_0, A_1 are disjoint subsets of Y such that $\overline{A_0} \cap \overline{A_1}$ contains a nonempty open subset U of M^{2^κ}. We can assume that U is a canonical neighborhood of M^{2^κ} that depends on a finite set $s \subset 2^\kappa$. Then there is an even successor ordinal $\gamma < 2^\kappa$ such that $s \subset \gamma$ and $\{x^{\gamma^-}_\eta\colon \eta \in A^0_\gamma\} = A_0$ and $\{x^{\gamma^-}_\eta\colon \eta \in A^1_\gamma\} = A_1$. It then follows from subcase 1 of case I that $\{x^\gamma_\eta \restriction \gamma\colon \eta \in A^0_\gamma\}$ is a nowhere dense subset of M^γ. Hence, A_0 is a nowhere dense subset of Y by (2_{2^κ}), a contradiction.

Prove (ii). Let B be a nowhere dense subset of Y and $a \in Z$, $a \in \overline{B}$. Since M is separable, there is an open set $U \subset M^{2^\kappa} \setminus B$ that depends on a countable index set $s \subset 2^\kappa$ and that is dense in $M^{2^\kappa} \setminus B$. The projection $B \restriction \gamma'$ is nowhere dense in $M^{\gamma'}$ for every γ' that contains s. Furthermore, there is an odd successor ordinal γ, $\gamma' \leqslant \gamma < 2^\kappa$, such that $a_\gamma = a$, and $\{x_\eta^{\gamma^-}: \eta \in B_\gamma\} = B$. Now by subcase 2 of case II, $a_\gamma{}^\gamma \restriction \gamma \notin \overline{(\{x_\eta^\gamma: \eta \in B_\gamma\}) \restriction \gamma}$, therefore $a \notin \overline{B}$ by (2_{2^κ}). This is a contradiction.

Now let B be a nowhere dense subset of Y and $a \in Y \setminus B$, $a \in \overline{B}$. Similarly to the previous paragraph, there is an odd successor ordinal $\gamma < 2^\kappa$ such that $B \restriction \gamma$ is a nowhere dense subset of 2^γ, $a \restriction \gamma \notin B \restriction \gamma$, and $x_{a_\gamma}^\gamma = a$, $B_\gamma = B$. Then by subcase 1 of case II, $x_{a_\gamma}^\gamma \restriction \gamma \notin \overline{(\{x_\eta^\gamma: \eta \in B_\gamma\}) \restriction \gamma}$, therefore $a \notin \overline{B}$ by (2_{2^κ}). A contradiction.

Prove (iii). Let A_1, A_2 be disjoint discrete subsets of Y. Similar to the proof of property (ii), there is an ordinal $\gamma' < 2^\kappa$ such that $A_1 \restriction \gamma'$ and $A_2 \restriction \gamma'$ are disjoint nowhere dense subsets of $M^{\gamma'}$. Then, there is an even successor ordinal γ, $\gamma' \leqslant \gamma < 2^\kappa$, such that $\{x_\eta^{\gamma^-}: \eta \in A_\gamma^0\} = A_0$ and $\{x_\eta^{\gamma^-}: \eta \in A_\gamma^1\} = A_1$. By subcase 2 of case I, there is a continuous function $f: X \to [0,1]$ that depends on the first γ coordinates and assumes constant values 0 and 1 on sets $\{x_\eta^\gamma \restriction \gamma: \eta \in A_\gamma^0\}$ and $\{x_\eta^\gamma \restriction \gamma: \eta \in A_\gamma^1\}$, respectively. The proof is complete. □

Corollary 4. *Let M be a separable metric space consisting of more than one point. For each infinite κ, there is a dense submaximal normal (hence perfectly normal) subset $X \subseteq M^{2^\kappa}$ of cardinality κ.*

Proof. Take $Z = \emptyset$ in Theorem 3 and use the fact that every normal ccc submaximal space is perfectly normal. (Indeed, every ccc submaximal space is perfect: to see this, suppose that U is open in such a space. Let $\{U_n: n \in \omega\}$ be a maximal disjoint family of open sets such that each $\overline{U_n} \subseteq U$. Since $\bigcup\{\overline{U_n}: n \in \omega\}$ covers all but a nowhere dense (hence closed) subset of U, U is an F_σ.) □

In the recent paper [4] D.P. Baturov studied perfectly normal dense subspaces of products of separable metric spaces. Theorem 3 generalizes two results from this paper. One is the particular case of $2^\kappa = 2^\omega$ of Theorem 3. The other is the case $\kappa = 2^\omega$ of the following corollary.

Corollary 5. *Let κ be an infinite cardinal and let M be a separable metric space consisting of more than one point. There is a perfectly normal dense submaximal subspace of M^{2^κ} that is strongly σ-discrete (is a union of countably many of its discrete closed subsets).*

Proof. Take $Z = \emptyset$ in Theorem 3 and use that X_0 has a countable projection $\{x_n: n \in \omega\}$ on the 0th coordinate. The resulting subset Y of M^{2^κ} will be submaximal and dense in M^{2^κ}. Therefore, $\pi_0^{-1}(x_n)$ will be a discrete and closed subset of Y for every $n \in \omega$. □

Corollary 6. *There is a separable submaximal not realcompact space of cardinality ω_1.*

Proof. In I^{2^ω}, take Z to be a convergent ω_1-sequence. E.g., for each $\alpha < \omega_1 \leqslant 2^\omega$, take a_α to be the point that is 0 at all coordinates $\beta \leqslant \alpha$ and is equal to 1 at all coordinates $> \alpha$. Then Z converges to the constant 0 function in the co-countable sense. The space $Y \cup Z$ given by Theorem 2 is submaximal (since Z is discrete) and, since any continuous $f: Y \cup Z \to [0,1]$ is determined by countably many coordinates, any zero set $Z' \subseteq Z$ is either countable or co-countable. Thus the filter of co-countable subsets of Z is a free zero-set ultrafilter with the countable intersection property. □

3. Another submaximal not realcompact space in ZFC

A construction previously considered by Levy and Porter [10] also can be used to provide a submaximal not realcompact space and a maximal not realcompact spaces in ZFC. For completeness sake we give a complete description of the spaces. The first example is obtained by blowing up the isolated points of a Ψ-space using a submaximal space. The maximal example, presented in the next section, is similar but requires a bit more work.

Given an almost disjoint family A, and a submaximal space $X \subseteq 2^{2^{\aleph_0}}$, we define a Tychonoff topology on $A \cup (\omega \times X)$.

We will need that X admits a remote filter. I.e., a family $\mathcal{U}$ of nonempty open subsets of X with the following properties

(1) $\mathcal{U}$ is a filter of open sets.

(2) For any dense (hence open) subset D of X there is $U \in \mathcal{U}$ such that $U \subseteq D$.
(3) For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $\overline{V} \subseteq U$.

The existence of such a family of open sets in an arbitrary (Tychonoff) topological space is equivalent to the existence of a remote point in $\beta X \setminus X$ ($p \in \beta X \setminus X$ is *remote* if p is not in the closure of any nowhere dense subset of X). We do not know the answer to the following question

Question 1. Does every (countable) submaximal Tychonoff space have a remote point?

Nonetheless, it is not hard to show that the $X \subseteq I^{2^\omega}$ constructed in Theorem 3 has a remote filter. Indeed, let $c \in [0,1]^{2^\omega}$ be the constant function with constant value 0.5 (any constant value different than 0 or 1 suffices by property (iii) of the proof). Let

$$\mathcal{U} = \left\{U \cap X\colon\ U \subseteq [0,1]^{2^\omega} \text{ is open and } c \in U\right\}.$$

Then $\mathcal{U}$ satisfies (1), (2) and (3).

Given a mad family A we define the topology on $A \cup (\omega \times X)$ as follows. The set $\omega \times X$ will be an open subspace where $\omega \times X$ is the direct sum $\bigoplus_n \{n\} \times X$ of ω copies of X. And a point $a \in A$ will have a neighborhoods of the form $\{a\} \cup \bigcup\{\{n\} \times U_n\colon\ n \in a \setminus F\}$ where F is finite, and each $U_n \in \mathcal{U}$. We will denote this space $Y = Y(A, X, \mathcal{U})$. We need to prove:

Claim 7. *For any almost disjoint family A and any submaximal X with a remote filter of open sets $\mathcal{U}$, the space $Y = Y(A, X, \mathcal{U})$ is Tychonoff and submaximal. Moreover, if $\Psi(A)$ is not realcompact, then neither is Y.*

Proof. Let D be a dense subset of $Y = Y(A, X, \mathcal{U})$. Then $D \cap \{n\} \times X$ is dense in $\{n\} \times X$ for each $n \in \omega$, hence open. Therefore, $D \cap \omega \times X$ is open. Suppose that $a \in D \cap A$. Then, for each $n \in a$, by remoteness of $\mathcal{U}$, we may choose $U_n \in \mathcal{U}$ such that $\{n\} \times U_n \subseteq D \cap \{n\} \times X$. This shows that a is in the interior of D. So D is open. Hence Y is submaximal.

It is easy to see that (3) implies that Y is regular. In the case that X is countable, Y is Tychonoff follows from the fact that any regular, locally countable space is Tychonoff.

In the case that X is not countable, suppose that $B \subseteq Y$ is a closed set and $a_0 \in A \setminus B$, then there is a neighborhood

$$U = \{a_0\} \cup \left\{\{n\} \times U_n\colon\ n \in a_0 \setminus F\right\}$$

of a_0 in Y which misses B. Let $r \in \beta X \setminus X$ be the point which corresponds to the remote filter $\mathcal{U}$. Then $r \notin cl_{\beta X}(B \cap \{n\} \times X)$ for all $n \in a_0 \setminus F$. Hence there is a continuous function $f_n : \beta X \to [0,1]$ such that $f_n(r) = 0$ and $f_n(B \cap \{n\} \times X) \subseteq \{1\}$ for all $n \in a_0 \setminus F$. Define $f : Y \to [0,1]$ by $f(a_0) = 0$, $f(a) = 1$ for all $a \in A \setminus \{a_0\}$, $f(n,x) = f_n(x)$ for all $n \in a_0 \setminus F$ and all $x \in X$ and $f(n,x) = 1$ for all $n \notin a_0 \setminus F$ and all $x \in X$. Then f is a continuous function which witnesses that a_0 and B are completely separated.

Finally to see that Y is not realcompact, for each n let $p_n \in \beta Y$ be a limit of the filter

$$\left\{\{n\} \times U\colon\ U \in \mathcal{U}\right\}.$$

Let $Z = \{p_n\colon\ n \in \omega\} \cup Y$.

The subspace $\{p_n\colon\ n \in \omega\} \cup A$ is a closed subspace of Z homeomorphic to $\Psi(A)$, thus Z is not realcompact. Therefore, there is a $p \in \beta Z \setminus Z = \beta Y \setminus Z$ such that any G_δ containing p meets Z. Since the space $Z \setminus Y$ is countable it must be that every G_δ containing p in fact meets Y. Thus Y is not realcompact. (This is essentially the proof that if Z is a subspace of βY obtained by adding to Y countably many points from $\beta Y \setminus Y$, then if Y is realcompact, so is Z.)[2] □

4. Maximal not realcompact examples

We now modify the constructions of the previous sections to give two constructions of a maximal not realcompact space. This answers a question of [1].

[2] We would like to thank Alan Dow for pointing out this proof.

Theorem 8 *(ZFC). There are separable maximal not realcompact spaces in ZFC.*

Proof. We give two constructions. For the first construction consider the example $Y \cup A$ of Corollary 6 with A the ω_1-sequence $\{a_\alpha\colon \alpha \in \omega_1\}$ that is convergent in the co-countable sense to $\vec{0}$.

To make $Y \cup A$ maximal, consider $Y \cup A$ as a subset of the Cantor cube C^{2^ω}. Let f denote a continuous map from the absolute of $A \cup Y$ onto C^{2^ω}, and let E be a subset of this absolute such that the restriction $f \restriction E$ is a one-to-one map onto $A \cup Y$. f will be irreducible, thus preimages of dense sets will be dense and images of nowhere dense sets will be nowhere dense. To ensure that E is not realcompact, we should be a little more careful about the preimages of A. Fix a single e in E that maps onto any given a in A. For a' in A, there's a unique e' in E such that the group action in 2^{2^ω} which takes a to a' will also take e to e' (as regular open ultrafilters on $Y \cap A$). See Lemma 3.4 of [3] where it is shown that in the absolute of 2^{2^ω} the family $E' = \{f^{-1}(a')\colon a' \in A\}$ will be converging in the co-countable sense to the point $f^{-1}(\vec{0})$. Since E is dense and extremally disconnected, it follows that βE is just its closure, hence z is a point of βE which does not have a G_δ disjoint from E.

For the second construction, we modify the topology on the Ψ-space example of Section 3. Let A be a mad family on ω, and for each $a \in A$ fix an ultrafilter $p_a \in a^*$. Let $D = \{p_a\colon a \in A\}$. Then D is discrete in $\beta\omega$. Let $Z(D) = \omega \cup D$ with the subspace topology inherited from $\beta\omega$. The role of the mad family A seems to be unimportant (see Question 2 below) and we only need to find a discrete $D \subseteq \omega^*$ such that this space $Z(D)$ is not realcompact.

We also need a maximal space X with a remote filter r. If a given maximal X does not have a remote filter, then for any point $x \in X$, the subspace $Y = X \setminus \{x\}$ is maximal and the neighborhood filter of x restricted to Y is a remote filter on Y.[3] Given X and r, we define $Y(Z(D), X, r)$ as in Section 3: The set $\omega \times X$ will be an open subspace where $\omega \times X$ is the direct sum $\bigoplus_n \{n\} \times X$ of ω copies of X. And a point $p \in D$ will have neighborhoods of the form $\{p\} \cup \bigcup\{\{n\} \times U_n\colon n \in x\}$ where $x \in p$ and each $U_n \in r$. The space $Y(Z(D), X, r)$ will always be regular (hence Tychonoff) as in the proof of Claim 7. We need to prove it maximal and not realcompact.

Claim 9. *Assuming X is maximal, so is $Y = Y(Z(D), X, r)$.*

Proof. Clearly Y has no isolated points. Suppose that τ is some stronger topology on Y. And let $V \in \tau$. By maximality of X, $V \cap \{n\} \times X$ is open in $\{n\} \times X$. So if V is not open in Y, there must be a $p \in V \cap D$ that is not in the interior of V. Consider the set $x = \{n\colon V \cap \{n\} \times X \in r\}$. If $x \in p$ then p would be in the interior of V. So $x \notin p$. But then, since maximal spaces are extremally disconnected and r is remote, for each $n \notin x$ we can find $U_n \in r$ such that $(\{n\} \times U_n) \cap V = \emptyset$. But then p is isolated in τ. □

Claim 10. *$Y = Y(Z(D), X, r)$ is not realcompact whenever $Z(D)$ is not realcompact.*

Proof. The proof is the same as for Claim 7: Adding a countable set of points to a realcompact subspace of βY preserves realcompactness. But $Z(D)$ can be realized as a closed subspace of $Y \cup \{r_n\colon n \in \omega\} \subseteq \beta Y$ where each r_n is the copy of the remote point r of $\{n\} \times X$. □

We are left with the question whether $Z(D)$ can fail to be realcompact for some discrete subset of ω^*. As in the previous construction, we may use a convergent ω_1 sequence in ω^*.

Claim 11. *If $D = \{p_\alpha\colon \alpha < \omega_1\}$ is a convergent sequence (without the limit point) in ω^*, then $D \cup \omega$ is not realcompact.*

Proof. In this case the co-countable filter on D in $Z(D)$ is a zero-set ultrafilter witnessing the failure of realcompactness. □

This construction raises some natural questions:

[3] Again, we thank Alan Dow for pointing out this construction and remarking that it is open whether or not every maximal space has a remote filter.

Question 2. Give a combinatorial characterization of those discrete subspace $D \subseteq \omega^*$, such that $Z(D)$ is not realcompact?

For example, if discreteness of D is witnessed by a mad family, is $Z(D)$ not realcompact? While it is possible to find a mad family A and choose $p_a \in a^*$ for each $a \in A$ such that $\omega \cup \{p_a\colon a \in A\}$ is realcompact, we do not know the answer to the following

Question 3. If A is mad, is there a choice of $p_a \in a^*$ for each $a \in A$ such that $\omega \cup \{p_a\colon a \in A\}$ is not realcompact?

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