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Countably compact spaces admitting full r-skeletons are proximal

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We provide simple characterizations of spaces admitting full r-skeletons, c-skeletons and q-skeletons, by using ω-monotone functions. We use this characterization to prove that every countably compact space admitting a full r-skeleton is proximal; furthermore the characterizations are used to show that the class of spaces admitting full c-skeletons is invariant under subspaces, disjoint topological sums and Σ-products, in addition to prove that the class of spaces admitting full q-skeletons is closed under extensions, continuous images and one-point Lindelöf extensions of disjoint topological sums. These characterizations also yield some positive results for products.

1. Introduction

One of the most important classes of compact spaces in Topology and Functional Analysis is the class of Corson compact spaces. Let $\mathbb{R}$ denote the real line, κ a cardinal, and $\mathbb{R}^\kappa$ the product of κ-many real lines. We consider the Σ-product

$$\Sigma\mathbb{R}^\kappa = \{x \in \mathbb{R}^\kappa : |\{\alpha \in \kappa : x_\alpha \neq 0\}| \leq \omega\}.$$

A compact space X is *Corson* if it is homeomorphic to a subspace of $\Sigma\mathbb{R}^\kappa$ for some cardinal κ. The study of Corson compact spaces is directly related with several important concepts in Topology and Functional Analysis, such as weakly Lindelöf determined Banach spaces [13, Chapters 3 and 5], r-skeletons [14, Chapter 19], and proximal spaces [7].

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The following result shows that a compact space is Corson if and only if it admits certain structure of retractions; full r-skeletons. This theorem follows from results of Bandlow [3] and Kubiś [15], and can be founded in [8].

Theorem 1.1. *A compact space X is Corson if and only if it admits a full r-skeleton.*

Some years later Casarrubias et al. provided another characterization of Corson compact spaces using a structure of closed subsets; full c-skeletons.

Theorem 1.2 *([5]). A compact space X is Corson if and only if it admits a full c-skeleton.*

This last characterization is useful to detect Corson compact subspaces of the spaces of continuous functions $C_p(X)$ for two reasons. First, full c-skeletons are inherited by arbitrary subspaces and so they can be identified in subspaces of $C_p(X)$ trough a dual property, a structure of $\mathbb{R}$-quotient functions; full q-skeletons. Indeed, the following two results were established in [5].

Theorem 1.3 *([5]). If X admits a full q-skeleton, then $C_p(X)$ admits a full c-skeleton.*

Theorem 1.4 *([5]). If X admits a full c-skeleton, then $C_p(X)$ admits a full q-skeleton.*

The other reason which makes full c-skeletons useful is that the class of spaces admitting a full q-skeleton is reasonably wide, as the following result shows (see [11] for the corresponding definitions).

Theorem 1.5 *([11]). The class of spaces admitting a full q-skeleton includes the following:*

(i) *spaces admitting a strong r-skeleton, particularly, monotonically retractable spaces and monotonically Sokolov spaces;*
(ii) *monotonically ω-stable spaces, in particular, pseudocompact and Lindelöf Σ-spaces.*

We do not know how much these classes can be extended, as currently, there is not known example of a space X for which every compact subspace of $C_p(X)$ is Corson but X does not admit a full q-skeleton.

The aim of this paper is to provide characterizations of r-skeletons, c-skeletons and q-skeletons by using ω-monotone functions. These characterizations are simple and in many cases let us simplify the use of these properties. We use them to prove that countably compact spaces admitting a full r-skeleton are proximal, also to determine the categorical behavior of the class of spaces admitting full c-skeletons and the class of spaces admitting full q-skeletons. We show that the class of spaces admitting full c-skeletons is invariant under subspaces, disjoint topological sums, and Σ-products; and show that the class of spaces admitting full q-skeletons is closed under extensions, continuous images and one-point Lindelöf extensions of disjoint topological sums. We also give some positive results for finite and infinite products. This development allows us to clarify the scope of the class of spaces admitting full q-skeletons while in the task of detecting Corson compact subspaces of $C_p(X)$-spaces. The results in this paper solve two open problems; [11, Question 4.10] and [5, Question 5.8].

2. Notation and preliminaries

In terminology and notation we follow [10] and [18]. All spaces in consideration are Tychonoff. By $\mathbb{R}$ we denote the real line together with its usual topology. The symbol κ denotes a cardinal. By ω we denote the first infinite ordinal, and $\mathfrak{c}$ denotes the cardinality of the continuum. Given a set X the families of all finite and countable subsets of X will be denoted as $[X]^{<\omega}$ and $[X]^{\leq\omega}$, respectively. The topology of a space X is denoted as $\tau(X)$. The closure of $A \subset X$ in X is denoted by $\mathrm{cl}_X(A)$ or simply $\mathrm{cl}(A)$.

We denote by $C_p(X, Y)$ the set of all continuous functions from X to Y endowed with the topology inherited from the product Y^X. If $A \subset C_p(X, Y)$ the *diagonal function* $\Delta_A : X \to Y^A$ of A is defined as $[\Delta_A(x)]_f = f(x)$ for each $x \in X$ and $f \in A$. Given $f : X \to Y$ the *weak topology* generated by f on X is $f^{-1}(\tau(Y))$. For $A \subset C_p(X, Y)$ the weak topology generated by A on X is the weak topology generated by Δ_A. Given $A \subset C_p(X, Y)$ and $B \subset X$, we say that A *separates the points of* B if for each pair of distinct points $x, y \in B$ there exists $f \in A$ such that $f(x) \neq f(y)$, that is, if $\Delta_A \restriction_B$ is injective. If $f : X \to Z$ and $g : X \to Y$ are continuous, we say that f *factorizes trough* g if there exists a continuous function $h : g(X) \to Z$ such that $f = h \circ g$. A function $f : X \to Y$ is $\mathbb{R}$-*quotient* if, for each $g : Y \to \mathbb{R}$, the continuity of $g \circ f$ implies the continuity of g. The space $C_p(X, \mathbb{R})$ will be denoted as $C_p(X)$. Given $f : X \to Y$ the *dual function* $f^* : C_p(Y) \to C_p(X)$ of f is defined by $f^*(g) = g \circ f$ for each $g \in C_p(Y)$. For $A \subset X$ the *restriction function* $\pi_A : C_p(X) \to C_p(A)$ is given by $\pi_A(f) = f \restriction_A$ for each $f \in C_p(X)$.

Along the paper, $(\Gamma, \leq)$ will denote an up-directed and σ-complete partially ordered set; when there is no possibility of confusion, we simply write Γ instead of $(\Gamma, \leq)$. Given a set X we will consider the set $[X]^{\leq\omega}$ partially ordered by $\subset$, which is up-directed and σ-complete.

Definition 2.1. A function $\phi : (\Gamma, \leq) \to (\Gamma', \preceq)$ between up-directed and σ-complete partially ordered sets is ω-*monotone* if satisfies the following:

(i) $\phi(s) \preceq \phi(t)$ whenever $s \leq t$;
(ii) if $\{s_n\}_{n\in\omega} \subset \Gamma$, $s_n \leq s_{n+1}$ for each $n \in \omega$, and $s = \sup_{n\in\omega} s_n$, then $\phi(s) = \sup_{n\in\omega} \phi(s_n)$.

Remark 2.2. Now we list without a proof some facts concerning ω-monotone functions that we will be using without reference along the text.

(i) The composition of ω-monotone functions is ω-monotone.
(ii) Given a function $f : X \to Y$ the function $\phi : [X]^{\leq\omega} \to [Y]^{\leq\omega}$ defined as $\phi(A) = f(A)$ is ω-monotone.
(iii) A function $\phi : [X]^{\leq\omega} \to [Y]^{\leq\omega}$ is ω-monotone if and only if there exists $\varphi : [X]^{<\omega} \to [Y]^{\leq\omega}$ such that $\phi(A) = \bigcup_{F\in[A]^{<\omega}} \varphi(F)$ for each $A \in [X]^{\leq\omega}$.
(iv) If $\varphi : [X]^{\leq\omega} \to [\kappa]^{\leq\omega}$ and $\phi_\alpha : [X]^{\leq\omega} \to [Y]^{\leq\omega}$ are ω-monotone for each $\alpha < \kappa$, then the function $\phi : [X]^{\leq\omega} \to [Y]^{\leq\omega}$ defined as $\phi(A) = \bigcup_{\alpha\in\varphi(A)} \phi_\alpha(A)$ is ω-monotone.

We will also need the following lemmas.

Lemma 2.3 *([11]). If $\varphi : [X]^{\leq\omega} \to [X]^{\leq\omega}$ is an ω-monotone function, then there exists and ω-monotone function $\phi : [X]^{\leq\omega} \to [X]^{\leq\omega}$ such that $A \subset \phi(A)$ and $\varphi(\phi(A)) \subset \phi(A)$ for each $A \in [X]^{\leq\omega}$.*

Lemma 2.4 *([11]). For each function $f : X \to \Gamma$ there exists an ω-monotone function $\phi : [X]^{\leq\omega} \to \Gamma$ such that $f(x) \leq \phi(\{x\})$ for each $x \in X$.*

The next two lemmas will be useful later. We will be using the fact that for a subset A of X and a continuous function $f : X \to Y$, the set $f(A)$ is dense in $f(X)$ if and only if A is a dense subspace of X in the weak topology generated by f.

Lemma 2.5. *If $A \subset C_p(X, Y)$, $D \subset X$ and $\Delta_F(D)$ is a dense subset of $\Delta_F(X)$ for each $F \in [A]^{<\omega}$, then $\Delta_A(D)$ is a dense subset of $\Delta_A(X)$.*

Proof. It suffices to show that D is a dense subset of X endowed with the weak topology generated by Δ_A. Let U be a nonempty basic open subset of X with the weak topology generated by Δ_A. Then there exists a nonempty finite set $F \in [A]^{<\omega}$ such that $U = \bigcap_{f\in F} f^{-1}(U_f)$, where U_f is open in Y for each $f \in F$. Observe that U is a nonempty open subset of X in the weak topology generated by F. By hypothesis D is a dense subset of X endowed with weak topology generated by F. Therefore $U \cap D \neq \emptyset$. □

Lemma 2.6. *If $A \subset C_p(X)$ and D_F is a dense subset of $\Delta_F^*(C_p(\Delta_F(X)))$ for each $F \in [A]^{<\omega}$, then $D = \bigcup_{F\in[A]^{<\omega}} D_F$ is a dense subset of $\Delta_A^*(C_p(\Delta_A(X)))$.*

Proof. Let U_A be a nonempty open set of the space $\Delta_A^*(C_p(\Delta_A(X)))$. Since the function $\Delta_A^* : C_p(\Delta_A(X)) \to C_p(X)$ is continuous, we can find a nonempty open set $V_A \subset C_p(\Delta_A(X))$ satisfying $\Delta_A^*(V_A) \subset U_A$. We can assume that

$$V_A = \{h \in C_p(\Delta_A(X)) : h(z_i) \in U_i, i = 1, \dots, n\}$$

for some set of distinct points $z_1, \dots, z_n \in \Delta_A(X)$, open sets $U_1, \dots, U_n \in \tau(\mathbb{R})$, and $n \in \mathbb{N}^+$. Choose $x_i \in X$ such that $\Delta_A(x_i) = z_i$ for each $i = 1, \dots, n$. Then we can find a finite set $F \subset A$ such that $\Delta_F(x_i) \neq \Delta_F(x_j)$ for each $i, j \in \{1, \dots, n\}$ with $i \neq j$. Let $y_i = \Delta_F(x_i)$ for $i = 1, \dots, n$. Since $\Delta_F(X)$ is a Tychonoff space we can find $f \in C_p(\Delta_F(X))$ such that $f(y_i) \in U_i$ for $i = 1, \dots, n$. It follows that $\Delta_F^*(f)(x_i) = f(\Delta_F(x_i)) \in U_i$ for $i = 1, \dots, n$. Hence the function $\Delta_F^*(f)$ belongs to the open subset

$$U_F = \{g \in \Delta_F^*(C_p(\Delta_F(X))) : g(z_i) \in U_i, i = 1, \dots, n\}$$

of $\Delta_F^*(C_p(\Delta_F(X)))$. Since D_F is a dense subset of $\Delta_F^*(C_p(\Delta_F(X)))$, we can find $g \in D_F \cap U_F \subset D \cap U_F$. Because $g \in \Delta_F^*(C_p(\Delta_F(X)))$, we can fix $h \in C_p(\Delta_F(X))$ such that $g = \Delta_F^*(h) = h \circ \Delta_F$. Note that $h(y_i) = h(\Delta_F(x_i)) = g(x_i) \in U_i$ for $i = 1, \dots, n$. Let $p : \Delta_A(X) \to \Delta_F(x)$ be the natural projection. Observe that the equality $\Delta_F = p \circ \Delta_A$ implies that $h(p(z_i)) = h(p(\Delta_A(x_i))) = h(\Delta_F(x_i)) = h(y_i) \in U_i$ for $i = 1, \dots, n$. It follows that $h \circ p \in V_A$ and so $g = \Delta_F^*(h) = (p \circ \Delta_A)^*(h) = \Delta_A^*(p^*(h)) = \Delta_A^*(h \circ p) \in \Delta_A^*(V_A) \subset U_A$. Therefore $g \in D \cap U_A$ and hence D is a dense subset of $\Delta_A^*(C_p(\Delta_A(X)))$. □

3. Spaces admitting full r-skeleton

The notion of r-skeleton was introduced by Kubiś and Michalewski in [16] where it was used to characterize Valdivia compact spaces. Using an ω-monotone function we give a characterization of countably compact spaces admitting full r-skeletons. Similar characterizations have been obtained using elementary substructures instead of ω-monotone functions in [3] when characterizing Corson compacta, in [15] when characterizing compact spaces with r-skeleton, and in [9] when characterizing countably compact spaces with full r-skeleton. Let us recall the definition of full r-skeleton.

Definition 3.1. A *full r-skeleton* in a space X is a family $\{r_s\}_{s\in\Gamma}$ of retractions on the space X satisfying:

(i) $r_s(X)$ has a countable network for each $s \in \Gamma$;
(ii) $r_s \circ r_t = r_t \circ r_s = r_s$ whenever $s \leq t$;
(iii) if $\{s_n\}_{n\in\omega} \subset \Gamma$, $s_n \leq s_{n+1}$ for each $n \in \omega$ and $t = \sup_{n\in\omega} s_n$, then $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for each $x \in X$; and
(iv) $X = \bigcup_{s\in\Gamma} r_s(X)$.

We are ready to prove the promised characterization.

Theorem 3.2. *A countably compact space X admits a full r-skeleton if and only if there exist an ω-monotone function $\phi : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\Delta_{\phi(A)}$ embeds* $\mathrm{cl}(A)$ *as a closed subspace in $\Delta_{\phi(A)}(X)$ for each $A \in [X]^{\leq\omega}$.*

Proof. For any $s \in \Gamma$ the space $r_s(X)$ is countably compact and has countable network weight, so it is a metrizable compact. Repeating the argument in [11, Proposition 2.6], we can assume that $\Gamma = [X]^{\leq\omega}$ and $A \subset r_A(X)$ for all $A \in \Gamma$. For each $n \in \omega$ let $p_n : \mathbb{R}^\omega \to \mathbb{R}$ be the projection of $\mathbb{R}^\omega$ onto the n-th coordinate. Choose $F \in [X]^{<\omega}$; since $r_F(X)$ is a metrizable compact, there exists an embedding $e_F : r_F(X) \to \mathbb{R}^\omega$. Let $M_F = \{p_n \circ e_F \circ r_F\}_{n\in\omega} \in [C_p(X)]^{\leq\omega}$ and observe that $\Delta_{M_F} \restriction_{r_F(X)} = e_F$ is an embedding. Consider the ω-monotone function $\phi : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ defined as $\phi(A) = \bigcup_{F\in[A]^{<\omega}} M_F$ for each $A \in [X]^{\leq\omega}$. Choose an arbitrary $A \in [X]^{\leq\omega}$; we will prove that $\Delta_{\phi(A)}$ embeds $\mathrm{cl}(A)$ as a closed subspace in $\Delta_{\phi(A)}(X)$. Observe that $A \subset r_A(X)$ implies $\mathrm{cl}(A) \subset r_A(X)$, hence $\mathrm{cl}(A)$ is compact and so it is sufficient to verify that $\Delta_{\phi(A)} \restriction_{r_A(X)}$ is injective. Choose a pair of distinct points $x, y \in r_A(X)$. The equalities $x = \sup_{F\in[A]^{<\omega}} r_F(x)$ and $y = \sup_{F\in[A]^{<\omega}} r_F(y)$ imply the existence of $F \in [A]^{<\omega}$ such that $r_F(x) \neq r_F(y)$. It follows that there exists $f \in M_F \subset \phi(A)$ such that $f(x) \neq f(y)$. This implies that $\Delta_{\phi(A)}(x) \neq \Delta_{\phi(A)}(y)$. Thus $\Delta_{\phi(A)} \restriction_{r_A(X)}$ is injective.

Now assume that there exist an ω-monotone function $\phi : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\Delta_{\phi(A)}$ embeds $\mathrm{cl}(A)$ as a closed subspace in $\Delta_{\phi(A)}(X)$ for each $A \in [X]^{\leq\omega}$. For each $G \in [C_p(X)]^{<\omega}$, since $\Delta_G(X)$ is separable, we can fix a set $D_G \in [X]^{\leq\omega}$ such that $\Delta_G(D_G)$ is a dense subset of $\Delta_G(X)$. Define $\varphi : [C_p(X)]^{\leq\omega} \to [X]^{\leq\omega}$ as $\varphi(B) = \bigcup_{G\in[B]^{<\omega}} D_G$ for each $B \in [C_p(X)]^{\leq\omega}$ and observe that this function is ω-monotone. Now, we can apply Lemma 2.3 to find an ω-monotone function $\delta : [X]^{\leq\omega} \to [X]^{\leq\omega}$ such that $A \subset \delta(A)$ and $\varphi(\phi(\delta(A))) \subset \delta(A)$ for each $A \in [X]^{\leq\omega}$. Choose $A \in [X]^{\leq\omega}$. For every $G \in [\phi(\delta(A))]^{<\omega}$ we have that $\Delta_G(D_G)$ is dense in $\Delta_G(X)$, and we know that $D_G \subset \varphi(\phi(\delta(A))) \subset \delta(A)$, so we conclude that $\Delta_G(\delta(A))$ is dense in $\Delta_G(X)$. Since this happens for any $G \in [\phi(\delta(A))]^{<\omega}$, we can apply Lemma 2.5 to conclude that $\Delta_{\phi(\delta(A))}(\delta(A))$ is dense in $\Delta_{\phi(\delta(A))}(X)$. Our hypothesis implies that $\Delta_{\phi(\delta(A))}(\mathrm{cl}(\delta(A)))$ is closed in $\Delta_{\phi(\delta(A))}(X)$, so $\Delta_{\phi(\delta(A))}(\mathrm{cl}(\delta(A))) = \Delta_{\phi(\delta(A))}(X)$. Therefore $\Delta_{\phi(\delta(A))} \restriction_{\mathrm{cl}(\delta(A))}$ is a homeomorphism and $r_A = (\Delta_{\phi(\delta(A))} \restriction_{\mathrm{cl}(\delta(A))})^{-1} \circ \Delta_{\phi(\delta(A))} : X \to \mathrm{cl}(\delta(A))$ is a retraction. We will verify that if $\Gamma = [X]^{\leq\omega}$, then $\{r_s\}_{s\in\Gamma}$ is a full r-skeleton in X with metrizable images.

(i) For each $s \in \Gamma$, $r_s(X)$ is homeomorphic to $\Delta_{\phi(\delta(s))}(X)$ and hence metrizable.
(ii) Note that if $s \in \Gamma$ and $x \in X$ then $r_s(x) = r_s(y)$ if and only if $\Delta_{\phi(\delta(s))}(x) = \Delta_{\phi(\delta(s))}(y)$. Choose $t \in \Gamma$ with $s \leq t$, it follows that $\delta(s) \subset \delta(t)$. On the one hand $r_s(X) = \mathrm{cl}(\delta(s)) \subset \mathrm{cl}(\delta(t)) = r_t(X)$ which implies $r_t(r_s(x)) = r_s(x)$ for all $x \in X$, that is, $r_s = r_t \circ r_s$. On the other hand, for each $x \in X$ the equality $r_t(x) = r_t(r_t(x))$ implies $\Delta_{\phi(\delta(t))}(x) = \Delta_{\phi(\delta(t))}(r_t(x))$, in particular $\Delta_{\phi(\delta(s))}(x) = \Delta_{\phi(\delta(s))}(r_t(x))$, and so $r_s(x) = r_s(r_t(x))$. Therefore $r_s = r_s \circ r_t$.
(iii) Let $\{s_n\}_{n\in\omega} \subset \Gamma$ such that $s_n \leq s_{n+1}$ for each $n \in \omega$ and let $t = \sup_{n\in\omega} s_n$. Choose an arbitrary $x \in X$. Given $f \in \phi(\delta(t)) = \bigcup_{n\in\omega} \phi(\delta(s_n))$ we can choose $N \in \omega$ such that $f \in \phi(\delta(s_n))$ for all $n \geq N$. If $n \geq N$, the equality $r_{s_n}(x) = r_{s_n}(r_{s_n}(x))$ implies $\Delta_{\phi(\delta(s_n))}(x) = \Delta_{\phi(\delta(s_n))}(r_s(x))$ and in particular $f(x) = f(r_{s_n}(x))$. Then $f(x) = \lim_{n\to\infty} f(r_{s_n}(x))$ and since $f \in \phi(\delta(t))$ was chosen arbitrarily we conclude that $\Delta_{\phi(\delta(t))}(x) = \lim_{n\to\infty} \Delta_{\phi(\delta(t))}(r_{s_n}(x))$. Therefore $r_t(x) = \lim_{n\to\infty} r_t(r_{s_n}(x)) = \lim_{n\to\infty} r_{s_n}(x)$.
(iv) $X = \bigcup_{s\in\Gamma} s \subset \bigcup_{s\in\Gamma} \delta(s) \subset \bigcup_{s\in\Gamma} \mathrm{cl}(\delta(s)) \subset \bigcup_{s\in\Gamma} r_s(X) \subset X$.

This finishes the proof. □

Observe that the proof of the second implication in Theorem 3.2 does not require countable compactness of X, this suggests that the second condition in this theorem is stronger than the existence of a full r-skeleton in an arbitrary space X.

We know that a compact space is Corson if and only if it admits a full r-skeleton [8, Theorem 3.11], if and only if it is proximal [7, Corollary 3.4]. Both notions are quite different, so it is natural to establish a general relation between spaces admitting r-skeletons and proximal spaces. Using the second condition in Theorem 3.2, we will show that every countably compact space admitting a full r-skeleton is proximal. Proximal spaces were introduced in [4], but we adopt the following equivalent definition obtained in [6].

Let X be a space and $\Delta = \{\langle x, x\rangle : x \in X\} \subset X^2$ its diagonal. An open neighborhood U of Δ in X^2 is symmetric if $U = \{\langle y, x\rangle : \langle x, y\rangle \in U\}$. Consider the family $\mathcal{U}_\Delta$ of all open symmetric neighborhoods U of Δ in X^2 such that there exists a sequence of neighborhoods $\{U_n\}_{n\in\omega}$ of Δ with $U_0 \subset U$ and $U_{n+1} \circ U_{n+1} \subset U_n$ for each $n \in \omega$. Observe that the family $\mathcal{U}_\Delta$ is closed under finite intersections. For each $U \in \mathcal{U}_\Delta$ and $x \in X$ we set $U[x] = \{y \in X : \langle x, y\rangle \in U\}$.

Definition 3.3. A space X is *proximal* if in the following two-player game there is a winning strategy for player 1. In inning 0, player 1 chooses $U_0 \in \mathcal{U}_\Delta$ and player 2 chooses $x_0 \in X$. In inning $n+1$, player 1 chooses $U_{n+1} \in \mathcal{U}_\Delta$ and player 2 chooses $x_{n+1} \in U_n[x_n]$. Then player 1 wins the game if either $\bigcap_{n\in\omega} U_n[x_n] = \emptyset$ or the sequence $\{x_n\}_{n\in\omega}$ converges.

If in the above definition player 1 must always ensure that $\{x_n\}_{n\in\omega}$ converges in X in order to win, then we say that X is *absolutely proximal.*

Theorem 3.4. *If X admits an ω-monotone function $\phi : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\Delta_{\phi(A)}$ embeds $\mathrm{cl}(A)$ as a closed subspace in $\Delta_{\phi(A)}(X)$ for each $A \in [X]^{\leq\omega}$, then X is proximal.*

Proof. Given $f \in C_p(X)$ and $n \in \omega$ consider the set

$$U(f, n) = \{\langle x, y\rangle \in X^2 : |f(x) - f(y)| < 1/2^n\},$$

and note that $U(f, n) \in \mathcal{U}_\Delta$. For each $A \in [X]^{\leq\omega}$ let $\{f_{A,n}\}_{n\in\omega}$ be an enumeration of $\phi(A)$. Let us play the proximal game on X.

- **Inning** 0**.** Make player 1 choose $U_0 = X$.
- **Inning** n**.** Assume that before this inning, player 2 has chosen a finite sequence $\{x_i\}_{i<n} \in X$. For each $i < n$ let $A_i = \{x_j\}_{j\leq i}$. Consider the set $U_n = \bigcap_{i<n} \bigcap_{j<n} U(f_{A_i,j}, n) \in \mathcal{U}_\Delta$ and make player 1 choose U_n in this inning.

We shall prove that this defines a winning strategy for player 1. Once the play finishes, consider the sequence $A = \{x_n\}_{n\in\omega}$ generated by player 2. Assume that $\bigcap_{n\in\omega} U_n[x_n] = \emptyset$ is not the case and fix a point $x \in \bigcap_{n\in\omega} U_n[x_n]$. Given $f \in \phi(A) = \phi(\bigcup_{n\in\omega} A_n) = \bigcup_{n\in\omega} \phi(A_n)$, there exists $i \in \omega$ such that $f \in \phi(A_i) = \{f_{A_i,j}\}_{j\in\omega}$; we then can choose $j \in \omega$ such that $f = f_{A_i,j}$. For each $n > \max\{i, j\}$ the election of U_n implies that $\langle x_n, x\rangle \in U_n \subset U(f, n)$, and so $|f(x_n) - f(x)| < 1/2^n$. It follows that $\{f(x_n)\}_{n\in\omega}$ converges to $f(x)$. Since $f \in \phi(A)$ is arbitrary, the sequence $\{\Delta_{\phi(A)}(x_n)\}_{n\in\omega}$ converges to $\Delta_{\phi(A)}(x)$. We know that $\Delta_{\phi(A)}$ embeds $\mathrm{cl}(A)$ as a closed subspace in $\Delta_{\phi(A)}(X)$, therefore the sequence $\{x_n\}_{n\in\omega}$ must converge to $(\Delta_{\phi(A)} \restriction_{\mathrm{cl}(A)})^{-1}(x)$. □

The following result is an immediate consequence of Theorems 3.2 and 3.4.

Corollary 3.5. *Every countably compact space admitting a full r-skeleton is proximal.*

4. Spaces admitting a full c-skeleton

Now we will characterize full c-skeletons by an ω-monotone function weaker than the function used to characterize full r-skeletons in countably compact spaces. This clarifies the relation between full r-skeletons and c-skeletons.

Definition 4.1. Given a space X let $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ be a family such that F_s is a closed subset of X and $\mathcal{B}_s$ is a countable collection of open subsets of X for each $s \in \Gamma$. We say that $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ is a *full c-skeleton* in X if:

(i) for each $s \in \Gamma$ the collection $\mathcal{B}_s$ is a base for a topology on X, and for the space $X_s = X$ endowed with this topology there exist a Tychonoff space Z_s and a continuous function $g_s : X_s \to Z_s$ such that $g_s \restriction_{F_s}$ is injective;
(ii) if $s, t \in \Gamma$ and $s \leq t$, then $F_s \subset F_t$;
(iii) the assignment $s \mapsto \mathcal{B}_s$ is ω-monotone; and
(iv) $X = \bigcup_{s\in\Gamma} F_s$.

Theorem 4.2. *A space X admits a full c-skeleton if and only if there exist an ω-monotone function $\sigma : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\sigma(A)$ separates the points of* $\mathrm{cl}(A)$ *for each $A \in [X]^{\leq\omega}$.*

Proof. Assume that $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ is a full c-skeleton in X and let $\mathcal{B} = \bigcup_{s\in\Gamma} \mathcal{B}_s$. For each $B, C \in \mathcal{B}$ if there exists $f \in C_p(X)$ such that $f(B) \cap f(C) = \emptyset$, then fix $f_{B,C} \in C_p(X)$ satisfying this property; otherwise let $f_{B,C} \equiv 0$. Define $\mu : \Gamma \to [C_p(X)]^{\leq\omega}$ by $\mu(s) = \{f_{B,C} : B, C \in \mathcal{B}_s\}$ for each $s \in \Gamma$. Observe that μ is an ω-monotone function. For each $x \in X = \bigcup_{s\in\Gamma} F_s$ we can fix $s_x \in \Gamma$ such that $x \in F_{s_x}$. By applying Lemma 2.4 we can find an ω-monotone function $\gamma : [X]^{\leq\omega} \to \Gamma$ such that $\gamma(\{x\}) \geq s_x$ for each $x \in X$. Define $\sigma : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ by $\sigma(A) = \bigcup_{G\in[A]^{<\omega}} \mu(\gamma(G))$ for each $A \in [X]^{\leq\omega}$. Note that this function also is ω-monotone. We will verify that σ satisfies the conditions of the theorem, i.e., given $A \in [X]^{\leq\omega}$, we shall prove that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$. Choose $x, y \in \mathrm{cl}(A)$. For each $z \in A$ we have that $s_z \leq \gamma(\{z\}) \leq \gamma(A)$ which implies $z \in F_{s_z} \subset F_{\gamma(A)}$. It follows that $A \subset F_{\gamma(A)}$ and, since F_s is closed in X, we deduce that $x, y \in \mathrm{cl}(A) \subset F_{\gamma(A)}$. By definition of a full c-skeleton there exists a Tychonoff space $Z_{\gamma(A)}$ and a continuous function $g_{\gamma(A)} : X_{\gamma(A)} \to Z_{\gamma(A)}$ such that $g_{\gamma(A)} \restriction_{F_{\gamma(A)}}$ is injective, and thus $g_{\gamma(A)}(x) \neq g_{\gamma(A)}(y)$. Fix $h \in C_p(Z_{\gamma(A)})$ such that $h(g_{\gamma(A)}(x)) \neq h(g_{\gamma(A)}(y))$. Let $f = h \circ g_{\gamma(A)} \in C_p(X_{\gamma(A)})$; since $\mathcal{B}_{\gamma(A)}$ is a base for $X_{\gamma(A)}$ and $f(x) \neq f(y)$, we can find neighborhoods $B, C \in \mathcal{B}_{\gamma(A)} \subset \mathcal{B}$ of x and y, respectively, such that $f(B) \cap f(C) = \emptyset$. Note that a function $f_{B,C}$ with this property has already been fixed. Since γ and the assignment $s \mapsto \mathcal{B}_s$ are ω-monotone, we have that $B_{\gamma(A)} = \bigcup_{G\in[A]^{<\omega}} \mathcal{B}_{\gamma(G)}$, and so we can choose $G \in [A]^{<\omega}$ such that $B, C \in \mathcal{B}_{\gamma(G)}$. It follows that $f_{B,C} \in \mu(\gamma(G)) \subset \sigma(A)$ and $f_{B,C}(x) \neq f_{B,C}(y)$. This shows that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$.

Now, suppose that there exist an ω-monotone function $\sigma : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$ for each $A \in [X]^{\leq\omega}$. Let $\mathcal{B}_{\mathbb{R}}$ be a countable base for $\mathbb{R}$ and for each $B \in [C_p(X)]^{\leq\omega}$ let $\mathcal{U}_B$ the family of all sets of the form $\bigcap_{f\in G} f^{-1}(U_f)$, where $G \in [B]^{<\omega}$ is nonempty and $U_f \in \mathcal{B}_{\mathbb{R}}$ for each $f \in G$. Observe that the assignment $B \mapsto \mathcal{U}_B$ is ω-monotone. Let $\Gamma = [X]^{\leq\omega}$ and for each $s \in \Gamma$ let $F_s = \mathrm{cl}(s)$ and $\mathcal{B}_s = \mathcal{U}_{\sigma(s)}$. We will prove that $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ is a full c-skeleton in X.

(i) For each $s \in \Gamma$, the family $\mathcal{B}_s$ is precisely the canonical base for the weak topology on X generated by the function $g_s = \Delta_{\sigma(s)} : X \to \mathbb{R}^{\sigma(s)}$, which by the hypothesis separates the points of F_s.
(ii) If $s, t \in \Gamma$ and $s \leq t$, then $F_s = \mathrm{cl}(s) \subset \mathrm{cl}(t) \subset F_t$.
(iii) It is clear the assignment $s \mapsto \mathcal{B}_s$ is ω-monotone.
(iv) $X = \bigcup_{s\in\Gamma} s \subset \bigcup_{s\in\Gamma} F_s \subset X$.

This finishes the proof. □

The following result solves Question 5.8 from [5].

Theorem 4.3. *A countably compact space X admits a full c-skeleton if and only if admits a full r-skeleton.*

Proof. Assume that X admits a full c-skeleton. By Theorem 4.2 there exists an ω-monotone function $\sigma : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\Delta_{\sigma(A)} \restriction_{\mathrm{cl}(A)}$ is injective for each $A \in [X]^{\leq\omega}$. Given $A \in [X]^{\leq\omega}$, the space $\mathrm{cl}(A)$ is countably compact and since every injective continuous function from a countably compact space onto a Fréchet-Urysohn space is a homeomorphism [18, Problem 140], then $\Delta_{\sigma(A)} \restriction_{\mathrm{cl}(A)}$ is an embedding. Since $\Delta_{\sigma(A)}(\mathrm{cl}(A))$ is a metrizable countably compact space, it is compact and hence closed in $\Delta_{\sigma(A)}(X)$. Theorem 3.2 implies that X admits a full r-skeleton.

For the other implication, assume that X admits a full r-skeleton. By Theorem 3.2 there exists an ω-monotone function $\phi : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\Delta_{\phi(A)}$ embeds $\mathrm{cl}(A)$ as a closed subspace in $\Delta_{\phi(A)}(X)$ for each $A \in [X]^{\leq\omega}$. In particular, $\phi(A)$ separates the points of $\mathrm{cl}(A)$, so by Theorem 4.2 the space X admits a full c-skeleton. □

In the following results we will use the characterization of c-skeletons from Theorem 4.2 to show that the class of spaces admitting full c-skeletons is closed under arbitrary subspaces, arbitrary disjoint topological sums, and Σ-products. The following result is stated in [11] without proof.

Theorem 4.4. *If X admits a full c-skeleton, then any subspace of X admits a full c-skeleton.*

Proof. Let Y be a subspace of X. We will prove that the natural restriction of a full c-skeleton in X is a full c-skeleton in Y. Since X admits a full c-skeleton, there exist an ω-monotone function $\sigma_X : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ such that $\Delta_{\sigma_X(A)} \restriction_{\mathrm{cl}_X(A)}$ is injective for each $A \in [X]^{\leq\omega}$. Consider the function $\sigma_Y : [Y]^{\leq\omega} \to [C_p(Y)]^{\leq\omega}$ defined as $\sigma_Y(A) = \pi_Y(\sigma_X(A))$ for each $A \in [C_p(Y)]^{\leq\omega}$. It is clear that this function is ω-monotone. To verify that the function σ_Y witnesses that Y admits a full c-skeleton fix $A \in [Y]^{\leq\omega}$, then $A \in [X]^{\leq\omega}$ and, since $\sigma_X(A)$ separates the points of $\mathrm{cl}_X(A)$, the function $\Delta_{\sigma_X(A)} \restriction_{\mathrm{cl}_X(A)}$ is injective. In particular $\Delta_{\sigma_X(A)} \restriction_{\mathrm{cl}_Y(A)}$ is injective. The equality $\Delta_{\sigma_Y(A)} = \Delta_{\sigma_X(A)} \restriction_Y$ implies that $\Delta_{\sigma_Y(A)} \restriction_{\mathrm{cl}_Y(A)} = \Delta_{\sigma_Y(A)} \restriction_{\mathrm{cl}_Y(A)}$ is also injective. □

Theorem 4.5. *If $\{X_\alpha\}_{\alpha<\kappa}$ is a family of spaces admitting a full c-skeleton, then the disjoint topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ admits full c-skeleton.*

Proof. Let $X = \bigoplus_{\alpha<\kappa} X_\alpha$ and choose $\alpha < \kappa$. Since X_α admits a full c-skeleton, there exists an ω-monotone function $\sigma_\alpha : [X_\alpha]^{\leq\omega} \to [C_p(X_\alpha)]^{\leq\omega}$ such that $\sigma_\alpha(A_\alpha)$ separates the points of $\mathrm{cl}_{X_\alpha}(A_\alpha)$ for each $A_\alpha \in [X_\alpha]^{\leq\omega}$. We can assume that $\sigma_\alpha(A_\alpha)$ is nonempty for each $A_\alpha \in [C_p(X_\alpha)]^{\leq\omega}$. Given $A \in [X]^{\leq\omega}$ set $A_\alpha = A \cap X_\alpha$ and define $\gamma : [X]^{\leq\omega} \to [\kappa]^{\leq\omega}$ by $\gamma(A) = \{\alpha \in \kappa : A_\alpha \neq \emptyset\}$; note that this function is ω-monotone. Fix an homeomorphism $h : \mathbb{R} \to (0,1)$ and choose $\alpha < \kappa$. For each $f_\alpha \in C_p(X_\alpha)$ let $\hat{f}_\alpha = (h \circ f_\alpha) \cup 0_\alpha : X \to \mathbb{R}$, where 0_α is the function identically zero with domain $X \setminus X_\alpha$. Observe that the function $\hat{\sigma}_\alpha : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ defined as $\hat{\sigma}_\alpha = \{\hat{f}_\alpha : f_\alpha \in \sigma_\alpha(A_\alpha)\}$ for each $A \in [X]^{\leq\omega}$ is ω-monotone. Finally consider the ω-monotone function $\sigma : [X]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ defined as $\sigma(A) = \bigcup_{\alpha\in\gamma(A)} \hat{\sigma}_\alpha(A_\alpha)$ for each $A \in [X_\alpha]^{\leq\omega}$. We will prove that the function σ witnesses that X admits a full c-skeleton. For a given $A \in [C_p(X)]^{\leq\omega}$, in order to verify that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$, choose $x, y \in \mathrm{cl}(A)$ and pick $\alpha, \beta < \kappa$ such that $x \in X_\alpha$ and $y \in X_\beta$. Observe that $x \in \mathrm{cl}_{X_\alpha}(A_\alpha)$, $y \in \mathrm{cl}_{X_\beta}(A_\beta)$ and $\alpha, \beta \in \gamma(A)$. If $\alpha = \beta$, then we can apply the fact that $\sigma_\alpha(A_\alpha)$ separates the points of $\mathrm{cl}_{X_\alpha}(A_\alpha)$ to find $f_\alpha \in \sigma_\alpha(A_\alpha)$ such that $f_\alpha(x) \neq f_\alpha(y)$. It follows that $\hat{f}_\alpha \in \sigma(A)$ and $\hat{f}_\alpha(x) = h(f_\alpha(x)) \neq h(f_\alpha(y)) = \hat{f}_\alpha(y)$. If $\alpha \neq \beta$, then for any $f_\alpha \in \sigma_\alpha(A_\alpha)$ we have that $\hat{f}_\alpha \in \sigma(A)$ and $\hat{f}_\alpha(x) = h(f_\alpha(x)) \neq 0 = \hat{f}_\alpha(y)$. This shows that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$. □

Theorem 4.6. *Every Σ-product of a family of spaces admitting a full c-skeleton admits a full c-skeleton.*

Proof. Let $\{X_\alpha\}_{\alpha<\kappa}$ be a family of spaces admitting a full c-skeleton and $X = \prod_{\alpha<\kappa} X_\alpha$. Fix a point $x \in X$ and consider the Σ-product

$$Y = \{y \in X : |\{\alpha \in \kappa : y_\alpha \neq x_\alpha\}| \leq \omega\}$$

of the family $\{X_\alpha\}_{\alpha<\kappa}$ with center at the point x.

For each $\alpha < \kappa$ the space X_α admits a full c-skeleton, so there exists an ω-monotone function $\sigma_\alpha : [X_\alpha]^{\leq\omega} \to [C_p(X_\alpha)]^{\leq\omega}$ such that $\sigma_\alpha(A_\alpha)$ separates the points of $\mathrm{cl}_{X_\alpha}(A_\alpha)$ for every $A_\alpha \in [X_\alpha]^{\leq\omega}$. For each $A \in [Y]^{\leq\omega}$ and $\alpha < \kappa$ let $A_\alpha = p_\alpha(A)$, where $p_\alpha : X \to X_\alpha$ is the natural projection. Given $S \in [\kappa]^{\leq\omega}$ let $Y_S = (\prod_{\alpha\in S} X_\alpha) \times \{p_{\kappa\setminus S}(x)\} \subset Y$. For each $y \in Y$ consider the set $S_y = \{\alpha < \kappa : y_\alpha \neq x_\alpha\}$; define $\gamma : [Y]^{\leq\omega} \to [\kappa]^{\leq\omega}$ by $\gamma(A) = \bigcup_{y\in A} S_y$ for each $A \in [Y]^{\leq\omega}$ and observe that this function is ω-monotone. Now consider the ω-monotone function $\sigma : [Y]^{\leq\omega} \to [C_p(Y)]^{\leq\omega}$ defined by $\sigma(A) = \bigcup_{\alpha\in\gamma(A)} q_\alpha^*(\sigma_\alpha(A_\alpha))$ for each $A \in [Y]^{\leq\omega}$; where $q_\alpha = p_\alpha \restriction_Y$ for each $\alpha < \kappa$. To see that this function witnesses that Y admits a full c-skeleton, given $A \in [Y]^{\leq\omega}$, we shall prove that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$. Choose two distinct points $y, z \in \mathrm{cl}(A)$. Since $\mathrm{cl}(A) \subset Y_{\gamma(A)}$, we can find $\alpha \in \gamma(A)$ such that $y_\alpha \neq x_\alpha$. The family $\sigma_\alpha(A_\alpha)$ separates the points of $\mathrm{cl}_{X_\alpha}(A_\alpha)$, so we can find $f_\alpha \in \sigma_\alpha(A_\alpha)$ such that $f_\alpha(x_\alpha) \neq f_\alpha(y_\alpha)$. It follows that $f_\alpha \circ q_\alpha = q_\alpha^*(f_\alpha) \in \sigma(A)$ and $f_\alpha \circ q_\alpha(x) \neq f_\alpha \circ q_\alpha(y)$. This shows that $\sigma(A)$ separates the points of $\mathrm{cl}(A)$. □

The following equivalences are known from the literature; we will give a proof using the results proved until now and the fact that proximal compact spaces are Corson.

Corollary 4.7. *For X compact, the following conditions are equivalent:*

(i) *The space X is Corson.*
(ii) *The space X admits a full c-skeleton.*
(iii) *The space X admits a full r-skeleton.*
(iv) *The space X is proximal.*

Proof. Since the real line admits a full c-skeleton, Theorems 4.6 and 4.4 imply that every Corson compact admits a full c-skeleton. Every compact space admitting a full c-skeleton admits a full r-skeleton because of Theorem 4.3. A compact space admitting a full r-skeleton is proximal because of Corollary 3.5. Finally, by [7, Corollary 3.4] every proximal compact space is Corson.

In relation with the previous result and Theorems 4.3 and 3.4, the following question arises naturally.

Question 4.8. *Is there a countably compact absolutely proximal space which does not admit a full r-skeleton?*

5. Spaces admitting a full q-skeleton

Now we will provide a characterization of full q-skeletons similar to the obtained for full r-skeletons and full c-skeletons, trough a ω-monotone function.

Definition 5.1. Given a space X, consider a family $\{(q_s, D_s)\}_{s\in\Gamma}$, where $q_s : X \to X_s$ is an $\mathbb{R}$-quotient function and D_s is a countable subset of X for each $s \in \Gamma$. We say that $\{(q_s, D_s)\}_{s\in\Gamma}$ is a *full q-skeleton* in X if:

(i) the set $q_s(D_s)$ is dense in X_s;
(ii) if $s,t \in \Gamma$ and $s \le t$, then there exists a continuous onto map $p_{t,s} : X_t \to X_s$ such that $q_s = p_{t,s} \circ q_t$;
(iii) the assignment $s \mapsto D_s$ is ω-monotone; and
(iv) $C_p(X) = \bigcup_{s\in\Gamma} q_s^*(C_p(X_s))$.

Theorem 5.2. *A space X admits a full q-skeleton if and only if there exist an ω-monotone function $\delta : [C_p(X)]^{\le\omega} \to [X]^{\le\omega}$ such that $\Delta_{\mathrm{cl}(A)}(\delta(A))$ is a dense subspace of $\Delta_{\mathrm{cl}(A)}(X)$ for each $A \in [C_p(X)]^{\le\omega}$.*

Proof. Assume that $\{(q_s, D_s) : s \in \Gamma\}$ is full q-skeleton in X. Since $C_p(X) = \bigcup_{s\in\Gamma} q_s^*(C_p(X_s))$, for each $f \in C_p(X)$ we can fix $s_f \in \Gamma$ such that $f \in q_{s_f}^*(C_p(X_{s_f}))$. By applying Lemma 2.4 we can find an ω-monotone function $\gamma : [C_p(X)]^{\le\omega} \to \Gamma$ such that $\gamma(\{f\}) \ge s_f$ for each $f \in C_p(X)$. Consider the ω-monotone function $\delta : [C_p(X)]^{\le\omega} \to [X]^{\le\omega}$ defined as $\delta(A) = D_{\gamma(A)}$ for each $A \in [C_p(X)]^{\le\omega}$. In order to verify the conditions in the theorem, choose $A \in [C_p(X)]^{\le\omega}$. Observe that if $f \in A$, then $s_f \le \gamma(\{f\}) \le \gamma(A)$ and so $f \in q_{s_f}^*(C_p(X_{s_f})) = (p_{\gamma(A),s_f} \circ q_{\gamma(A)})^*(C_p(X_{s_f})) = q_{\gamma(A)}^*(p_{\gamma(A),s_f}^*(C_p(X_{s_f}))) \subset q_{\gamma(A)}^*(C_p(X_{\gamma(A)}))$. It follows that $A \subset q_{\gamma(A)}^*(C_p(X_{\gamma(A)}))$. Since $q_{\gamma(A)}$ is an $\mathbb{R}$-quotient function, by [18, Problem 163 (iii)] the set $q_{\gamma(A)}^*(C_p(X_{\gamma(A)}))$ is closed in $C_p(X)$ and so $\mathrm{cl}(A) \subset q_{\gamma(A)}^*(C_p(X_{\gamma(A)}))$. Then for each $f \in \mathrm{cl}(A)$ we can choose $g_f \in C_p(X_{\gamma(A)})$ such that $f = q_{\gamma(A)}^*(g_f) = g_f \circ q_{\gamma(A)}$. Let $B = \{g_f\}_{f\in\mathrm{cl}(A)} \subset C_p(X_{\gamma(A)})$ and note that $\Delta_{\mathrm{cl}(A)} = \Delta_B \circ q_{\gamma(A)}$. By hypothesis $q_{\gamma(A)}(D_{\gamma(A)}) = q_{\gamma(A)}(\delta(A))$ is dense in $X_{\gamma(A)}$ and hence in $q_{\gamma(A)}(X)$, thus the continuity of Δ_B implies that $\Delta_B(q_{\gamma(A)}(\delta(A)))$ is dense in $\Delta_B(q_{\gamma(A)}(X))$. Therefore $\Delta_{\mathrm{cl}(A)}(\delta(A))$ is dense in $\Delta_{\mathrm{cl}(A)}(X)$.

Now, assume that $\delta : [C_p(X)]^{\le\omega} \to [X]^{\le\omega}$ is an ω-monotone function such that $\Delta_{\mathrm{cl}(A)}(\delta(A))$ is a dense subspace of $\Delta_{\mathrm{cl}(A)}(X)$ for each $A \in [C_p(X)]^{\le\omega}$. We will construct an ω-monotone function $\phi : [C_p(X)]^{\le\omega} \to [C_p(X)]^{\le\omega}$ such that $\Delta_{\mathrm{cl}(\phi(A))}$ is an $\mathbb{R}$-quotient function for each $A \in [C_p(X)]^{\le\omega}$. Choose $F \in [C_p(X)]^{<\omega}$ and let $X_F = \Delta_F(X)$. Since $d(C_p(X_F)) \le nw(C_p(X_F)) = nw(X_F) \le \omega$, and $\Delta_F^* : C_p(X_F) \to C_p(X)$ is an embedding, we can fix a countable dense subset F' of $\Delta_F^*(C_p(X_F))$. Consider the ω-monotone function $\varphi : [C_p(X)]^{\le\omega} \to [C_p(X)]^{\le\omega}$ defined as $\varphi(A) = \bigcup\{F' : F \in [A]^{<\omega}\}$ for each $A \in [C_p(X)]^{\le\omega}$. Given $A \in [C_p(X)]^{\le\omega}$ it follows from Lemma 2.6 that $\varphi(A)$ is a dense subset of $\Delta_A^*(C_p(X_A))$. Now, we can apply Lemma 2.3 to find an ω-monotone function $\phi : [C_p(X)]^{\le\omega} \to [C_p(X)]^{\le\omega}$ such that $A \subset \phi(A)$ and $\varphi(\phi(A)) \subset \phi(A)$ for each $A \in [C_p(X)]^{\le\omega}$. Given $A \in [C_p(X)]^{\le\omega}$, we know that $\varphi(\phi(A))$ is a dense subset of $\Delta_{\phi(A)}^*(C_p(X_{\phi(A)}))$. Since $\varphi(\phi(A)) \subset \phi(A)$, the set $\phi(A)$ also is a dense subset of $\Delta_{\phi(A)}^*(C_p(X_{\phi(A)}))$. We then can apply [11, Lemma 4.6] to see that $\Delta_{\mathrm{cl}(\phi(A))}$ is an $\mathbb{R}$-quotient function. Now let $\Gamma = [C_p(X)]^{\le\omega}$, and for each $s \in \Gamma$ consider the $\mathbb{R}$-quotient function $q_s = \Delta_{\mathrm{cl}(\phi(s))} : X \to \Delta_{\mathrm{cl}(\phi(s))}(X)$ and the countable set $D_s = \delta(\phi(s)) \subset X$. We will verify that $\{(q_s, D_s)\}_{s\in\Gamma}$ is a full q-skeleton in X.

(i) By hypothesis $q_s(D_s) = \Delta_{\mathrm{cl}(\phi(s))}(\delta(\phi(s)))$ is dense in $X_s = \Delta_{\mathrm{cl}(\phi(s))}(X)$.
(ii) If $s,t \in \Gamma$ and $s \le t$, then the natural projection $p_{t,s} : \Delta_{\mathrm{cl}(\phi(t))}(X) \to \Delta_{\mathrm{cl}(\phi(s))}(X)$ is a continuous onto map such that $q_s = p_{t,s} \circ q_t$.
(iii) The assignment $s \mapsto D_s$, being the composition of two ω-monotone functions, is ω-monotone.
(iv) $C_p(X) \subset \bigcup_{s\in\Gamma} s \subset \bigcup_{s\in\Gamma} \phi(s) \subset \bigcup_{s\in\Gamma} q_s^*(C_p(X_s)) \subset C_p(X)$.

This finishes the proof of the theorem. □

The following result states the most important property of full q-skeletons, which can be obtained as an immediate consequence of Theorems 1.3, 4.4 and Corollary 4.7.

Theorem 5.3 *([11]). If X admits a full q-skeleton, then every compact subspace of $C_p(X)$ is Corson.*

It is clear that every separable space admits a full q-skeleton. As was stated in Theorem 1.5 (ii), this also happens to Lindelöf Σ-spaces. It was proved in [17, example 3.15] that there exists a Lindelöf P-space X

such that $C_p(X)$ contains a compact subspace which is not Corson. So, Theorem 5.3 implies that Lindelöf P-spaces need not admit full q-skeletons.

Now we will use the characterization obtained in Theorem 5.2 to show that the class of spaces admitting a full q-skeleton is closed under extensions, continuous images, and countable unions. We will also give an example to show that full q-skeletons are not preserved under the product of two spaces.

Theorem 5.4. *If Y is dense in X and Y admits a full q-skeleton, then X admits a full q-skeleton.*

Proof. Since Y admits a full q-skeleton, there exists an ω-monotone function $\delta_Y : [C_p(Y)]^{\leq\omega} \to [Y]^{\leq\omega}$ such that $\Delta_{\mathrm{cl}(A)}(\delta_Y(A))$ is a dense subspace of $\Delta_{\mathrm{cl}(A)}(Y)$ for each $A \in [C_p(Y)]^{\leq\omega}$. Define $\delta_X : [C_p(X)]^{\leq\omega} \to [X]^{\leq\omega}$ as $\delta_X(B) = \delta_Y(\pi_Y(B))$ for each $B \in [C_p(X)]^{\leq\omega}$ and observe that this function is ω-monotone. We shall prove that this function witnesses that X admits a full q-skeleton. Fix $B \in [C_p(X)]^{\leq\omega}$; the continuity of $\pi_Y : C_p(X) \to C_p(Y)$ implies that $\pi_Y(\mathrm{cl}(B)) \subset \mathrm{cl}(\pi_Y(B))$. By hypothesis $\Delta_{\mathrm{cl}(\pi_Y(B))}(\delta_X(B)) = \Delta_{\mathrm{cl}(\pi_Y(B))}(\delta_Y(\pi_Y(B)))$ is dense in $\Delta_{\mathrm{cl}(\pi_Y(B))}(Y)$. Since $\pi_Y(\mathrm{cl}(B)) \subset \mathrm{cl}(\pi_Y(B))$, the function $\Delta_{\pi_Y(\mathrm{cl}(B))}$ factorizes trough $\Delta_{\mathrm{cl}(\pi_Y(B))}$, and so $\Delta_{\pi_Y(\mathrm{cl}(B))}(\delta_X(B))$ is dense in $\Delta_{\pi_Y(\mathrm{cl}(B))}(Y)$. From the equality $\Delta_{\pi_Y(\mathrm{cl}(B))} = \Delta_{\mathrm{cl}(B)} \restriction_Y$ we deduce that $\Delta_{\mathrm{cl}(B)}(\delta_X(B))$ is dense in $\Delta_{\mathrm{cl}(B)}(Y)$. Finally the density of Y in X implies that $\Delta_{\mathrm{cl}(B)}(Y)$ is dense in $\Delta_{\mathrm{cl}(B)}(X)$ and therefore $\Delta_{\mathrm{cl}(B)}(\delta_X(B))$ is dense in $\Delta_{\mathrm{cl}(B)}(X)$. □

The following result solves Question 4.10 from [11].

Theorem 5.5. *If X admits a full q-skeleton and Y is a continuous image of X, then Y admits a full q-skeleton.*

Proof. Let $f : X \to Y$ be a continuous onto function. Since X admits a full q-skeleton, there exists an ω-monotone function $\delta_X : [C_p(X)]^{\leq\omega} \to [X]^{\leq\omega}$ such that $\Delta_{\mathrm{cl}(A)}(\delta_X(A))$ is a dense subspace of $\Delta_{\mathrm{cl}(A)}(X)$ for each $A \in [C_p(X)]^{\leq\omega}$. Define $\delta_Y : [C_p(Y)]^{\leq\omega} \to [Y]^{\leq\omega}$ as $\delta_Y(B) = f(\delta_X(f^*(B)))$ for each $B \in [C_p(Y)]^{\leq\omega}$ and note that this function is ω-monotone. We will verify that this function witnesses that Y admits a full q-skeleton. Fix $B \in [C_p(Y)]^{\leq\omega}$, then the continuity of $f^* : C_p(Y) \to C_p(X)$ implies that $f^*(\mathrm{cl}(B)) \subset \mathrm{cl}(f^*(B))$. By hypothesis $\Delta_{\mathrm{cl}(f^*(B))}(\delta_X(f^*(B)))$ is dense in $\Delta_{\mathrm{cl}(f^*(B))}(X)$. Since $f^*(\mathrm{cl}(B)) \subset \mathrm{cl}(f^*(B))$, the function $\Delta_{f^*(\mathrm{cl}(B))}$ factorizes trough $\Delta_{\mathrm{cl}(f^*(B))}$ and so $\Delta_{f^*(\mathrm{cl}(B))}(\delta_X(f^*(B)))$ is dense in $\Delta_{f^*(\mathrm{cl}(B))}(X)$. From the equality $\Delta_{f^*(\mathrm{cl}(B))} = \Delta_{\mathrm{cl}(B)} \circ f$ we deduce that $\Delta_{f^*(\mathrm{cl}(B))}(\delta_X(f^*(B))) = \Delta_{\mathrm{cl}(B)}(f(\delta_X(f^*(B)))) = \Delta_{\mathrm{cl}(B)}(\delta_Y(B))$ and $\Delta_{f^*(\mathrm{cl}(B))}(X) = \Delta_{\mathrm{cl}(B)}(f(X)) = \Delta_{\mathrm{cl}(B)}(Y)$. Therefore $\Delta_{\mathrm{cl}(B)}(\delta_Y(B))$ is dense in $\Delta_{\mathrm{cl}(B)}(Y)$. □

The following result shows a case when full q-skeletons are inherited by a subspace; however we do not know a more satisfactory result in this direction.

Corollary 5.6. *If X admits a full q-skeleton and Y is open and closed in X, then Y admits a full q-skeleton.*

Let us consider the following construction. If $\{X_\alpha\}_{\alpha<\kappa}$ is a family of spaces, consider the disjoint topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ and the space $\bigoplus_{\alpha<\kappa} X_\alpha \cup \{\infty\}$ obtained from the sum by adding the point ∞ which we can assume that does not belong to $\bigoplus_{\alpha<\kappa} X_\alpha$ and has for a basis the family of all sets of the form $U_S = X \setminus \bigoplus_{\alpha\in S} X_\alpha$, where $S \in [\kappa]^{\leq\omega}$.

Theorem 5.7. *If $\{X_\alpha\}_{\alpha<\kappa}$ is a family of spaces admitting a full q-skeleton, then the space $X = \bigoplus_{\alpha<\kappa} X_\alpha \cup \{\infty\}$ admits a full q-skeleton.*

Proof. Given $\alpha < \kappa$, since X_α admits a full q-skeleton, there exists an ω-monotone function $\delta_\alpha : [C_p(X_\alpha)]^{\leq\omega} \to [X_\alpha]^{\leq\omega}$ such that $\Delta_{\mathrm{cl}(A_\alpha)}(\delta_\alpha(A_\alpha))$ is a dense subspace of $\Delta_{\mathrm{cl}(A_\alpha)}(X)$ for each $A_\alpha \in$

$[C_p(X_\alpha)]^{\leq\omega}$. Observe that for every $f \in C_p(X)$ there exists a set $S_f \in [\kappa]^{\leq\omega}$ such that f is constant on U_{S_f}. Consider the ω-monotone function $\sigma : [C_p(X)]^{\leq\omega} \to [\kappa]^{\leq\omega}$ defined as $\sigma(A) = \bigcup_{f\in A} S_f$ for each $A \in [C_p(X)]^{\leq\omega}$. For every $A \in [C_p(X)]^{\leq\omega}$ let $A_\alpha = \pi_{X_\alpha}(A)$. Now consider the ω-monotone function $\delta : [C_p(X)]^{\leq\omega} \to [X]^{\leq\omega}$ defined as $\delta(A) = \{\infty\} \cup \bigcup_{\alpha\in\sigma(A)} \delta_\alpha(A_\alpha)$ for each $A \in [C_p(X)]^{\leq\omega}$. We will verify that this function witnesses that X admits a full q-skeleton. Fix a set $A \in [C_p(X)]^{\leq\omega}$. Observe that each function $f \in A$ is constant on $U_{\sigma(A)}$ and, as a consequence, every function $g \in \mathrm{cl}(A)$ is constant on $U_{\sigma(A)}$. So $\Delta_{\mathrm{cl}(A)}$ is constant on $U_{\sigma(A)}$. Since $\infty \in \delta(A) \cap U_{\sigma(A)}$, we conclude that $\Delta_{\mathrm{cl}(A)}(\delta(A))$ is dense in $\Delta_{\mathrm{cl}(A)}(U_{\sigma(A)})$. Now fix $\alpha \in \sigma(A)$. We know that $\Delta_{\mathrm{cl}(A_\alpha)}(\delta_\alpha(A_\alpha))$ is a dense subspace of $\Delta_{\mathrm{cl}(A_\alpha)}(X_\alpha)$. It follows from continuity that $\pi_{X_\alpha}(\mathrm{cl}(A)) \subset \mathrm{cl}(A_\alpha)$, and so $\Delta_{\pi_{X_\alpha}(\mathrm{cl}(A))}$ factorizes trough $\Delta_{\mathrm{cl}(A_\alpha)}$. Then $\Delta_{\pi_{X_\alpha}(\mathrm{cl}(A))}(\delta_\alpha(A_\alpha))$ is a dense subspace of $\Delta_{\pi_{X_\alpha}(\mathrm{cl}(A))}(X_\alpha)$. The equality $\Delta_{\pi_{X_\alpha}(\mathrm{cl}(A))} = \Delta_{\mathrm{cl}(A)} \restriction_{X_\alpha}$ implies that $\Delta_{\mathrm{cl}(A)}(\delta_\alpha(A_\alpha))$ is a dense subspace of $\Delta_{\mathrm{cl}(A)}(X_\alpha)$. Finally, from the above and the definition of $\delta(A)$, we conclude that $\Delta_{\mathrm{cl}(A)}(\delta(A))$ is a dense subspace of $\Delta_{\mathrm{cl}(A)}(U_{\sigma(A)}) \cup \bigcup_{\alpha\in\sigma(A)} \Delta_{\mathrm{cl}(A)}(X_\alpha) = \Delta_{\mathrm{cl}(A)}(X)$. □

Corollary 5.8. *If $X = \bigcup_{n\in\omega} X_n$ and each X_n admits a full q-skeleton, then X admits a full q-skeleton.*

Proof. By Theorem 5.7 the space $\bigoplus_{n\in\omega} X_n \cup \{\infty\}$ admits a full q-skeleton. Since the disjoint topological sum $\bigoplus_{n\in\omega} X_n$ is closed and open in $\bigoplus_{n\in\omega} X_n \cup \{\infty\}$, Corollary 5.6 implies that $\bigoplus_{n\in\omega} X_n$ admits a full q-skeleton. The space X, being a continuous image of $\bigoplus_{n\in\omega} X_n$, admits a full q-skeleton because of Theorem 5.5. □

Proposition 5.9. *If X contains an uncountable open and closed discrete subspace, then X does not admit a full q-skeleton.*

Proof. By Corollary 5.6 it is sufficient to prove the case in which X is itself discrete and has cardinality ω_1. In this case $C_p(X) = \mathbb{R}^X$ contains a countable dense subspace A. It follows that $\mathrm{cl}(A) = \mathbb{R}^X$ and so the weak topology generated by $\mathrm{cl}(A)$ in X is the discrete topology. Since the discrete topology in X is not separable, there is no set $D \in [X]^{\leq\omega}$ such that $\Delta_{\mathrm{cl}(A)}(D)$ is dense in $\Delta_{\mathrm{cl}(A)}(X)$. Therefore X does not admit a full q-skeleton. □

There are pseudocompact spaces admitting uncountable closed discrete subspaces; since by Theorem 1.5 (ii) every pseudocompact space admits a full q-skeleton, Proposition 5.9 implies that full q-skeletons are not inherited by closed subspaces. Likewise, the following example shows that the product of two arbitrary spaces admitting a full q-skeleton not necessarily admits a full q-skeleton.

Example 5.10. There exists a space X admitting a full q-skeleton whose square does not admit a full q-skeleton.

Proof. Consider the ordinal $\mathfrak{c}$ with the discrete topology. Let $\{B_1, B_2\}$ be a partition of $\mathfrak{c}$ in sets of cardinality $\mathfrak{c}$ and let $b : B_1 \to B_2$ be a bijection. Put $f = b \cup b^{-1} : \mathfrak{c} \to \mathfrak{c}$ and let $g : \beta\mathfrak{c} \to \beta\mathfrak{c}$ be the continuous extension of f. Note that $f = f^{-1}$ and $g = g^{-1}$. If $\hat{b} : \mathrm{cl}(B_1) \to \mathrm{cl}(B_2)$ is the continuous extension of b, then $g = \hat{b} \cup \hat{b}^{-1}$, and this shows that g has no fixed point. Choose two enumerations $\{A_\alpha\}_{\alpha\in B_1}$ and $\{A_\alpha\}_{\alpha\in B_2}$ of all the countable infinite subsets of $\mathfrak{c}$, where each element appears $\mathfrak{c}$-many times. Define a sequence $\{x_\alpha\}_{\alpha<\mathfrak{c}} \subset \beta\mathfrak{c}$ recursively as follows: assume that we have defined $\{x_\beta\}_{\beta<\alpha}$ for some $\alpha < \mathfrak{c}$; since every infinite closed subset of $\beta\mathfrak{c}$ has cardinality at least $2^{\mathfrak{c}}$ [12, Corollary 9.12], we can fix a point $x_\alpha \in \beta\mathfrak{c} \setminus (\mathfrak{c} \cup \{g(x_\beta)\}_{\beta<\alpha} \cup \{g^{-1}(x_\beta)\}_{\beta<\alpha})$, such that $x_\alpha \in \mathrm{cl}(A_\alpha)$ whenever $\alpha \in B_1$, and satisfying that $x_\alpha \in \mathrm{cl}(\{x_\beta\}_{\beta\in A_\alpha})$ whenever $\alpha \in B_2$ and $A_\alpha \subset \alpha$. Once the construction have finished take $X = \mathfrak{c} \cup \{x_\alpha\}_{\alpha<\mathfrak{c}}$. We will verify that X satisfies the required properties.

We claim that X is countably compact. Indeed, choose a countable infinite subset A of X. To show that A contains an accumulation point in X, it is sufficient to consider the cases $A \subset \mathfrak{c}$ and $A \subset \{x_\alpha\}_{\alpha<\mathfrak{c}}$. If

$A \subset \mathfrak{c}$ then $A = A_\alpha$ for some $\alpha < \mathfrak{c}$ and so in step α the point $x_\alpha \in X$ has been chosen as an accumulation point of A. If $A \subset \{x_\alpha\}_{\alpha<\mathfrak{c}}$ then $A = \{x_\beta\}_{\beta \in A_\alpha}$ for $\mathfrak{c}$-many ordinals $\alpha < \mathfrak{c}$ and so we can fix $\alpha < \mathfrak{c}$ such that $A = \{x_\beta\}_{\beta\in A_\alpha}$ and $A_\alpha \subset \alpha$; it follows that in step α the point $x_\alpha \in X$ has been chosen as an accumulation point of A. This shows that X is countably compact and in particular pseudocompact. Then [5, Corollary 4.8] implies that X admits a full q-skeleton. On the other hand, observe that $F = \{(\alpha, f(\alpha))\}_{\alpha<\mathfrak{c}}$, the graph of f, is discrete, open and closed in $X \times X$. Indeed, if $G = \{(x, g(x))\}_{x\in\beta\mathfrak{c}}$ is the graph of g, then the continuity of g implies that G is closed in $\beta\mathfrak{c} \times \beta\mathfrak{c}$. It follows from the fact that g has no fixed point, the equality $g = g^{-1}$, and the construction of X, that $F = G \cap (X \times X)$. Hence F is closed in $X \times X$. Moreover, the set F is discrete and open in $X \times X$ since the contention $F \subset \mathfrak{c} \times \mathfrak{c}$ implies that all points of f are isolated. Finally, since F is discrete, open and closed in $X \times X$, we can apply Proposition 5.9 to see that $X \times X$ admits no full q-skeleton. □

Observe that the technique used in the above example can not be used in $\beta\omega$ since the product of separable spaces is separable and every separable space admits a full q-skeleton.

6. Product of spaces admitting full q-skeletons

We will show now that under some additional conditions the product of two spaces admitting a full q-skeleton also admits a full q-skeleton; when the corresponding product is Lindelöf. Let us introduce some notation.

Given two spaces X and Y, and given $f \in C_p(X \times Y)$, for each $x \in X$ we will consider the continuous function $f_x \in C_p(Y)$ defined as $f_x(y) = f(x, y)$ for each $y \in Y$, and for each $y \in Y$ we will consider the continuous function $f_y \in C_p(X)$ defined as $f_y(x) = f(x, y)$ for each $x \in X$.

Lemma 6.1. *If $X \times Y$ is Lindelöf and $f \in C_p(X \times Y)$, then there exists a countable set $D \subset X$ such that $\{f_x\}_{x\in X} \subset \mathrm{cl}(\{f_x\}_{x\in D})$.*

Proof. For each $n \in \omega$ let $\mathcal{U}_n$ be a cover of $\mathbb{R}$ consisting of open sets with diameter less that $1/2^n$. Let $\mathcal{V}_n$ be a cover of $X \times Y$ consisting of nonempty basic open sets which refines $f^{-1}(\mathcal{U}_n)$. Since the space $X \times Y$ is Lindelöf, there exists a countable subcover $\mathcal{B}_n$ of $\mathcal{V}_n$. Observe that $f(\mathcal{B}_n)$ refines $\mathcal{U}_n$ and so all elements of $f(\mathcal{B}_n)$ have diameter less that $1/2^n$. Let $p_X : X \times Y \to X$ be the projection onto the first coordinate and let $\mathcal{F}_n$ be the family of all nonempty finite intersections of members of $p_X(\mathcal{B}_n)$. Choose a countable set $D_n \subset X$ which intersects each member of $\mathcal{F}_n$ and let $D = \bigcup_{n\in\omega} D_n$. Given an arbitrary point $x_0 \in X$, we shall prove that $f_{x_0} \in \mathrm{cl}(\{f_x\}_{x\in D})$. Consider an arbitrary basic open neighborhood

$$U = \{g \in C_p(Y) : \forall(y \in F)(|f_{x_0}(y) - g(y)| < 1/2^n)\}$$

of f_{x_0}, where $F \subset Y$ is a nonempty finite set and $n \in \omega$. For each $y \in F$, since $\mathcal{B}_n$ is a cover of $X \times Y$, we can choose $B_y \in \mathcal{B}_n$ such that $\langle x_0, y\rangle \in B_y$. Note that $x_0 \in \bigcap_{y\in F} p_X(B_y)$ and this implies that $\bigcap_{y\in F} p_X(B_y) \in \mathcal{F}_n$. By construction there exists a point $x \in D_n \cap \bigcap_{y\in F} p_X(B_y)$; it is clear that $x \in D$. Given $y \in F$, note that $\langle x_0, y\rangle, \langle x, y\rangle \in B_y$ and, since $f(B_y)$ has diameter less that $1/2^n$, we have that $|f_{x_0}(y) - f_x(y)| = |f(\langle x_0, y\rangle) - f(\langle x, y\rangle)| < 1/2^n$. It follows that $f_x \in U$. Since $x \in D$ and U is arbitrary, we conclude that $f_{x_0} \in \mathrm{cl}(\{f_x\}_{x\in D})$. □

Theorem 6.2. *Assume that both spaces X and Y admit a full q-skeleton; if $X \times Y$ is Lindelöf, then $X \times Y$ admits a full q-skeleton.*

Proof. Let $Z = X \times Y$. Since both spaces X and Y admit a full q-skeleton, there exists an ω-monotone function $\delta_X : [C_p(X)]^{\leq\omega} \to [X]^{\leq\omega}$ such that for each $B \in [C_p(X)]^{\leq\omega}$ the set $\delta_X(B)$ is a dense subspace of X endowed with the weak topology generated by $\mathrm{cl}(B)$, and also there exists an ω-monotone function $\delta_Y : [C_p(Y)]^{\leq\omega} \to [Y]^{\leq\omega}$ such that $\delta_Y(C)$ is a dense subspace of Y endowed with the weak topology generated by $\mathrm{cl}(C)$ for each $C \in [C_p(Y)]^{\leq\omega}$. Given $f \in C_p(Z)$ by Lemma 6.1 there exist a set $D_f \in [X]^{\leq\omega}$ such that $\{f_x\}_{x\in X} \subset \mathrm{cl}(\{f_x\}_{x\in D_f})$ and a set $E_f \in [Y]^{\leq\omega}$ such that $\{f_y\}_{y\in X} \subset \mathrm{cl}(\{f_y\}_{y\in E_f})$. Define the functions $\sigma_X : [C_p(Z)]^{\leq\omega} \to [X]^{\leq\omega}$ and $\sigma_Y : [C_p(Z)]^{\leq\omega} \to [Y]^{\leq\omega}$ as $\sigma_X(A) = \bigcup_{f\in A} D_f$ and $\sigma_Y(A) = \bigcup_{f\in A} E_f$, for each $A \in [C_p(Z)]^{\leq\omega}$, and note that these functions are ω-monotone. Consider the functions $\phi_X : [C_p(Z)]^{\leq\omega} \to [C_p(X)]^{\leq\omega}$ and $\phi_Y : [C_p(Z)]^{\leq\omega} \to [C_p(Y)]^{\leq\omega}$ defined as $\phi_X(A) = \{f_y : y \in \sigma_Y(A) \text{ and } f \in A\}$ and $\phi_Y(A) = \{f_x : x \in \sigma_X(A) \text{ and } f \in A\}$ for each $A \in [X]^{\leq\omega}$. Note that ϕ_X and ϕ_Y also are ω-monotone functions. Finally consider the ω-monotone function $\delta_Z : [C_p(Z)]^{\leq\omega} \to [Z]^{\leq\omega}$ given by $\delta_Z(A) = \delta_X(\phi_X(A)) \times \delta_Y(\phi_Y(A))$ for each $A \in [C_p(Z)]^{\leq\omega}$. We shall prove that this function witnesses that Z admits a full q-skeleton.

Choose $A \in [C_p(Z)]^{\leq\omega}$. We will verify that $\delta_Z(A)$ is a dense subspace of Z endowed with the weak topology generated by $\mathrm{cl}(A)$. Fix a nonempty set $U = \bigcap_{f\in F} f^{-1}(U_f)$ of Z with the weak topology generated by $\mathrm{cl}(A)$, where $F \subset \mathrm{cl}(A)$ is a nonempty finite set and U_f is open in $\mathbb{R}$ for each $f \in F$. Choose a point $z_0 = \langle x_0, y_0\rangle \in U$. Identifying X with the subspace $X \times \{y_0\}$ of Z and using the fact that $F \subset \mathrm{cl}(A)$, we can see that the continuity of the restriction function implies that $\{f_{y_0}\}_{f\in F} \subset \mathrm{cl}(\{f_{y_0}\}_{f\in A})$. Note that $\{f_{y_0}\}_{f\in A} \subset \bigcup_{f\in A} \mathrm{cl}(\{f_y\}_{y\in E_f}) \subset \mathrm{cl}(\bigcup_{f\in A}\{f_y\}_{y\in E_f})) \subset \mathrm{cl}(\phi_X(A))$ which implies $\{f_{y_0}\}_{f\in F} \subset \mathrm{cl}(\{f_{y_0}\}_{f\in A}) \subset \mathrm{cl}(\phi_X(A))$. It follows that $U_X = \bigcap_{f\in F} f_{y_0}^{-1}(U_f)$ is an open subset of X in the weak topology generated by $\mathrm{cl}(\phi_X(A))$. Note that U_X is nonempty since $x_0 \in U_X$. We know that $\delta_X(\phi_X(A))$ is a dense subspace of X in the weak topology generated by $\mathrm{cl}(\phi_X(A))$, so we can fix a point $x_1 \in \delta_X(\phi_X(A)) \cap U_X$. Observe that $\langle x_1, y_0\rangle \in U$. Now we will apply a similar argument to find $y_1 \in Y$ such that $\langle x_1, y_1\rangle \in \delta_Z(A) \cap U$. Since $F \subset \mathrm{cl}(A)$, and identifying Y with the subspace $\{x_1\} \times Y$ of Z, the continuity of the restriction function implies that $\{f_{x_1}\}_{f\in F} \subset \mathrm{cl}(\{f_{x_1}\}_{f\in A})$. The contention $\{f_{x_1}\}_{f\in A} \subset \bigcup_{f\in A} \mathrm{cl}(\{f_x\}_{x\in D_f}) \subset \mathrm{cl}(\bigcup_{f\in A}\{f_x\}_{x\in D_f})) \subset \mathrm{cl}(\phi_Y(A))$ implies that $\{f_{x_1}\}_{f\in F} \subset \mathrm{cl}(\phi_Y(A))$. Hence $U_Y = \bigcap_{f\in F} f_{x_1}^{-1}(U_f)$ is an open subset of Y in the weak topology generated by $\mathrm{cl}(\phi_Y(A))$. Note that U_Y is nonempty since $y_0 \in U_Y$. We know that $\delta_Y(\phi_Y(A))$ is a dense subspace of Y in the weak topology generated by $\mathrm{cl}(\phi_Y(A))$, so we can fix a point $y_1 \in \delta_Y(\phi_Y(A)) \cap U_Y$. Observe that $\langle x_1, y_1\rangle \in \delta_Z(A) \cap U$. Therefore $\delta_Z(A)$ is a dense subspace of Z endowed with the weak topology generated by $\mathrm{cl}(A)$. □

In Question 5.20 from [1] Arhangel'skii asked whether the class of all spaces X satisfying that every compact subspace of $C_p(X)$ is Corson is closed under products with compact spaces. In relation with this question and Theorem 6.2, the following questions seem to be interesting.

Question 6.3. *Is it true that if X admits a full q-skeleton and K is compact, then $X \times K$ admits a full q-skeleton?*

Question 6.4. *Let $\mathcal{K}$ be the class of all spaces X such that every compact subspace of $C_p(X)$ is Corson. Suppose that K is a compact space and X is a Lindelöf space such that $X \in \mathcal{K}$. Is it true that $X \times K \in \mathcal{K}$?*

The following result shows that, unlike the general case, the property of admitting a full q-skeleton is preserved under arbitrary products in the class of C_p-spaces.

Theorem 6.5. *If $\{C_p(X_\alpha)\}_{\alpha<\kappa}$ is a family of spaces admitting a full q-skeleton, then the space $\prod_{\alpha<\kappa} C_p(X_\alpha)$ admits a full q-skeleton.*

Proof. Choose $\alpha < \kappa$. The fact that $C_p(X_\alpha)$ admits a full q-skeleton implies, because of Theorem 1.3, that $C_p(C_p(X_\alpha))$ admits a full c-skeleton. Then Theorem 4.4 implies that X_α also admits a full c-skeleton. We then can apply Theorem 4.5 to see that the disjoint topological sum $\bigoplus_{\alpha<\kappa} X_\alpha$ admits a full c-skeleton. Now, an application of Theorem 1.4 implies that $C_p(\bigoplus_{\alpha<\kappa} X_\alpha)$ admits a full q-skeleton. It is well known [18, Problem 114] that the spaces $C_p(\bigoplus_{\alpha<\kappa} X_\alpha)$ and $\prod_{\alpha<\kappa} C_p(X_\alpha)$ are homeomorphic. Therefore the product $\prod_{\alpha<\kappa} C_p(X_\alpha)$ admits a full q-skeleton. □

The above theorem suggests that in some other cases spaces admitting a full q-skeleton have a chance to be productive; the following theorem shows one of these cases.

Theorem 6.6. *Let $\{X_\alpha\}_{\alpha<\kappa}$ be a family of spaces such that $\prod_{\alpha\in F} X_\alpha$ is Lindelöf and admits a full q-skeleton for each finite set $F \subset \kappa$. If Y is a σ-product in $X = \prod_{\alpha<\kappa} X_\alpha$ and each $f \in C_p(Y)$ can be factorized through $p_{S_f} \restriction_Y$ for some countable set $S_f \subset \kappa$, then Y admits a full q-skeleton.*

Proof. Fix a point $x \in X$ such that

$$Y = \{y \in X : |\{\alpha \in \kappa : y_\alpha \neq x_\alpha\}| < \omega\}$$

is the σ-product of the family $\{X_\alpha\}_{\alpha<\kappa}$ with center at the point x. Choose $F \in [\kappa]^{<\omega}$ and let $Y_F = (\prod_{\alpha\in F} X_\alpha) \times \{p_{\kappa\setminus F}(x)\} \subset Y$. The space Y_F is homeomorphic to X_F and hence admits a full q-skeleton, that is, there exists an ω-monotone function $\delta_F : [C_p(Y_F)]^{\leq\omega} \to [Y_F]^{\leq\omega}$ such that, for each $B \in [C_p(Y_F)]^{\leq\omega}$, the set $\delta_F(B)$ is a dense subspace of Y_F endowed with the weak topology generated by $\mathrm{cl}(B)$. Consider the function $\sigma : [C_p(Y)]^{\leq\omega} \to [\kappa]^{\leq\omega}$ defined by $\sigma(A) = \bigcup_{f\in A} S_f$ for each $A \in [C_p(Y)]^{\leq\omega}$ and note that this function is ω-monotone. Now consider the ω-monotone function $\delta : [C_p(Y)]^{\leq\omega} \to [Y]^{\leq\omega}$ defined by $\delta(A) = \bigcup_{F\in[\sigma(A)]^{\leq\omega}} \delta_F(\pi_{Y_F}(A))$ for each $A \in [C_p(Y)]^{\leq\omega}$. We will verify that this function witnesses that Y admits a full q-skeleton.

Given $A \in [C_p(Y)]^{\leq\omega}$, we shall prove that $\delta(A)$ is a dense subspace of Y in the weak topology generated by $\mathrm{cl}(A)$. Let $U = \bigcap_{f\in G} f^{-1}(U_f)$ be a nonempty basic open subset of Y in the weak topology generated by $\mathrm{cl}(A)$, that is, where G is a nonempty finite subset of $\mathrm{cl}(A)$ and U_f is open in $\mathbb{R}$ for each $f \in G$. We know that each function $f \in A$ factorizes through $p_{S_f} \restriction_Y$, thus the contention $S_f \subset \sigma(A)$ implies that each function $f \in A$ factorizes through $p_{\sigma(A)} \restriction_Y$. Since the function $p_{\sigma(A)} \restriction_Y$ is open and in particular $\mathbb{R}$-quotient, we can apply [18, Problem 163 (iii)] to see that each function $f \in \mathrm{cl}(A)$ factorizes through $p_{\sigma(A)} \restriction_Y$. Fix a point $y \in U$. Since each function $f \in \mathrm{cl}(A)$ factorizes through $p_{\sigma(A)} \restriction_Y$, we can assume that $F = \{\alpha \in \kappa : y_\alpha \neq x_\alpha\} \subset \sigma(A)$. Hence $F \in [\sigma(A)]^{<\omega}$ and $y \in Y_F$. Since $\pi_{Y_F}(\mathrm{cl}(A)) \subset \mathrm{cl}(\pi_{Y_F}(A))$, then $U_F = U \cap Y_F = \bigcap_{f\in G}(f \restriction_{Y_F})^{-1}(U_f)$ is a nonempty open subset of Y_F in the weak topology generated by $\mathrm{cl}(\pi_{Y_F}(A))$. The set $\delta_F(\pi_{Y_F}(A))$ is a dense subspace of Y_F endowed with the weak topology generated by $\mathrm{cl}(\pi_{Y_F}(A))$ and so there exists a point $z \in \delta_F(\pi_{Y_F}(A)) \cap U_F \subset \delta(A) \cap U$. Therefore $\delta(A)$ is a dense subspace of Y endowed with the weak topology generated by $\mathrm{cl}(A)$. □

We can apply Theorem 6.6 in combination with Theorem 5.4 to obtain the following corollary.

Corollary 6.7. *If $\{X_n\}_{n\in\omega}$ is a family of spaces such that $\prod_{i\leq n} X_i$ admits a full q-skeleton for each $n \in \omega$, then $\prod_{n\in\omega} X_n$ admits a full q-skeleton.*

It was proved in Theorem 2.19 from [1] that if Y is any subspace of the product of a family of Lindelöf Σ-spaces which contains a σ-product of this family, then any compact subspace of $C_p(Y)$ is Fréchet-Urysohn and ω-monolithic. Since Corson compact spaces are Fréchet-Urysohn and ω-monolithic, and because of Theorems 5.3 and 1.5 (ii), this result admits the following generalization.

Corollary 6.8. *Let $\{X_\alpha\}_{\alpha<\kappa}$ be a family of spaces such that $\prod_{\alpha\in F} X_\alpha$ is Lindelöf and admits a full q-skeleton for each finite set $F \subset \kappa$. Let Z be a dense subspace of $\prod_{\alpha<\kappa} X_\alpha$ containing a σ-product, then the space Z admits a full q-skeleton.*

Proof. Let Y be a σ-product in $\prod_{\alpha<\kappa} X_\alpha$. We can apply [18, Problem 298] and [2, Corollary 1.6.45] to see that each $f \in C_p(Y)$ can be factorized trough $p_{A_f} \restriction_Y$ for some countable set $A_f \subset \kappa$. It follows from Theorem 6.6 that Y admits a full q-skeleton. Finally, we can apply Theorem 5.4 to conclude that Z admits a full q-skeleton. $\square$

In particular, any product of Lindelöf Σ-spaces and any product of separable spaces admit a full q-skeleton.

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