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A SMALL DOWKER SPACE FROM A CLUB-GUESSING PRINCIPLE

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Abstract. We present the construction of a new Dowker space from a special type of club-guessing ladder system. These types of guessing principles have previously been used to construct spaces consistent with $\mathsf{MA}+\neg\mathsf{CH}$. Thus, this construction may shed light on whether $\mathsf{MA}+\neg\mathsf{CH}$ is consistent with the existence of a Dowker space of size $\aleph_1$.

1. Introduction

C. H. Dowker proved that a product $X \times [0,1]$ is normal if and only if X is normal and countable paracompact [3]. Subsequently, any normal space X that has non-normal product with the closed unit interval has come to be called a *Dowker space*. Whether ZFC implies there is a Dowker space of cardinality ω_1 is a particular and important instance of the general "small Dowker space question." Indeed, it is not known whether $\mathsf{MA}+\neg\mathsf{CH}$ or PFA implies there are no Dowker spaces of size $\aleph_1$ [7, Problem 10].

Although $\mathsf{MA}+\neg\mathsf{CH}$ decides a great deal about structures of size ω_1, there are a number of counterexamples to this general principle. For example, the existence of either a first countable S-space [1] or

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perfectly normal not realcompact spaces of size $\aleph_1$ [4], [5] is independent of MA+¬CH. The latter result used a club-guessing ladder system to construct a perfectly normal not realcompact space followed by a consistency result to show that the club guessing principle was consistent with MA+¬CH. The main motivation of this paper was to try to use similar techniques to obtain a Dowker space of size $\aleph_1$ in a model where MA+¬CH holds. While we have not been able to obtain such a space, we do present a construction of an example using a club-guessing principle not previously seen in the literature, and we conjecture that similar techniques should lead to an example consistent with MA+¬CH or even PFA.

It should be remarked that it would even be interesting to show that this example is consistent with some weak form of Martin's Axiom. Teruyuki Yorioka [8] has recently shown that generalizations of one of Rudin's constructions cannot be Dowker assuming $\mathcal{K}_2(\mathrm{r}ec)$, and it would be interesting in general to understand what kind of small Dowker spaces there are in the model for Katětov's problem (where $\mathcal{K}_2(\mathrm{r}ec)$ holds) [6].

In section 2 we introduce the guessing principle, explain how to construct our space, and introduce the properties of the guessing sequence that imply the space is Dowker. In section 3 we present the construction of the required guessing sequence assuming $V = L$. In section 4 we show how to modify the space to make it locally compact and first countable.

2. Building the space from the guessing sequence

A *ladder system on* ω_1 is a sequence $\overrightarrow{E} = \langle E_\alpha : \alpha \in \mathrm{Lim}\,(\omega_1)\rangle$ such that each $E_\alpha \subseteq \alpha$ is of order type ω and unbounded in α.

If $\overrightarrow{E}$ is a ladder system on ω_1, then we define a topology τ on ω_1 associated with $\overrightarrow{E}$ by declaring final segments of its elements E_α as weak neighborhoods of α; that is, a set $U \subseteq \omega_1$ is defined to be τ-open if for each $\alpha \in U$, there exists some $\beta < \alpha$ such that $E_\alpha \setminus \beta \subseteq U$. One can easily check that this is a topology, and that with this topology, ω_1 is a regular space. Of course, we cannot expect to always get a normal topology. However, if $\overrightarrow{E}$ has good guessing properties, then (ω_1, τ) can be normal. In order to state the necessary lemma, we need some definitions. We say that $\overrightarrow{E}$ is *strong club guessing* if for every club $C \subseteq \omega_1$, there is a club $K \subseteq \omega_1$

so that $\alpha \in K \Rightarrow (\exists\beta < \alpha)\,(E_\alpha \setminus \beta \subseteq C)$, and $\overrightarrow{E}$ is *2-stationary hitting* if for every pair of stationary sets $S, T \subseteq \omega_1$, there is some $\gamma \in \omega_1$ such that both $E_\gamma \cap S$ and $E_\gamma \cap T$ are cofinal in γ. It can be proven than if $\overrightarrow{E}$ is 2-stationary hitting, then there are stationarily many $\gamma < \omega_1$ witnessing that property for every pair of stationary sets. This type of guessing principle was used in [5] to construct a perfectly normal not realcompact space; please refer to this paper for a proof of the following lemma.

Lemma 1. *If $\overrightarrow{E}$ is strong club guessing and 2-stationary hitting, then (ω_1, τ) is normal.*

Recall that Dowker proved in [3] that a normal space X is countably paracompact if whenever $\{D_n\}_{n\in\omega}$ is a decreasing family of closed subsets of X whose intersection is empty, there exists a family $\{U_n\}_{n\in\omega}$ of open sets which has empty intersection and, for each $n \in \omega$, $U_n \supseteq D_n$. We start by considering a countable partition of ω_1 into stationary subsets $\{S_n : n \in \omega\}$, and we will make the union of the first n many stationary sets open. The complement of that union will be the closed set D_n witnessing that countable paracompactness fails. So, we need that the elements of the ladder system $\overrightarrow{E}$ "look back"; that is, $\alpha \in S_n$ must give us $E_\alpha \subseteq \bigcup_{k=0}^{n} S_k$.

Proposition 2. *Suppose $\{S_n : n \in \omega\}$ is a partition of ω_1 into stationary subsets and $\overrightarrow{E}$ is a strong club guessing ladder system such that $\alpha \in S_n \Rightarrow E_\alpha \subseteq \bigcup_{k=0}^{n} S_k$. Moreover, assume that for each n and for each pair of stationary sets $A, B \subseteq \bigcup_{i<n} S_i$, there is $\alpha \in S_n$ such that $E_\alpha \cap A$ and $E_\alpha \cap B$ are infinite. Then, letting τ be the topology associated with $\overrightarrow{E}$, the space (ω_1, τ) is a Dowker space.*

Proof: It is easy to see that the hypotheses imply that the ladder system is 2-stationary hitting, so by Lemma 1 the space is normal. To see that the space is not countably paracompact, let $D_n = \omega_1 \setminus \bigcup_{i<n} S_i$. Then $\{D_n : n \in \omega\}$ is a decreasing sequence of closed sets with empty intersection. Now suppose that $\langle W_n : n \in \omega\rangle$ is a sequence of τ-open subsets of ω_1 such that for each $n \in \omega$, $W_n \supseteq D_n$. Suppose further that $\bigcap_{n\in\omega} W_n = \emptyset$. Then $(\forall\alpha \in \omega_1)\,(\exists n \in \omega)\,(\alpha \notin W_n)$, and thus, for some fixed $m \in \omega$, there is a stationary set $X \subseteq \omega_1 \setminus W_m$. As W_m is open, $\mathrm{cl}_\tau(X) \subseteq \omega_1 \setminus W_m$.

However, there must be $\delta \in S_{m+1}$ such that

$$\delta \in \mathrm{cl}_\tau(X) \subseteq \omega_1 \setminus W_m,$$

which contradicts that $S_{m+1} \subseteq W_m$. Hence, the space is not countably paracompact. $\Box$

Before we present the construction of a ladder system satisfying the hypotheses of the previous proposition, we make a few observations about spaces constructed from ladder systems. Recall that a space X is *scattered* if and only if every non-empty subset contains an isolated point. One can also define $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$ —the set of limit points, and $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ in case α is a limit ordinal. Then X is a scattered space if and only if there is some ordinal α such that $X^{(\alpha)} = \emptyset$. If such an α exists, then we say that the *scattered height* of X is the least α for which $X^{(\alpha)} = \emptyset$.

Lemma 3. *If $\overrightarrow{E}$ is a strong club guessing ladder system and τ is the topology on ω_1 described above, then (ω_1, τ) has uncountable scattered height.*

Proof: To see that (ω_1, τ) has uncountable scattered height, it will suffice to show, by induction for every $\alpha < \omega_1$, that $(\omega_1)^{(\alpha)}$ contains a club subset of ω_1.

Suppose $\alpha < \omega_1$ is a limit ordinal. Choose $\beta_n \nearrow \alpha$. Then $(\omega_1)^{(\alpha)} = \bigcap_{n=0}^{\infty} (\omega_1)^{(\beta_n)}$. Thus, we need only to make sure that for successor stages at least a club of limit points is preserved. But that is easy as $\overrightarrow{E}$ is a strong club guessing sequence. $\Box$

By a result in [5], we know that using ladder systems $\overrightarrow{E}$ we cannot hope for an example consistent with $\mathsf{MA}+\neg\mathsf{CH}$ unless we use "longer" ladders. That is, we will need that for at least a stationary set of α's the E_α's have order type α. We refer to this kind of sequence as a guessing sequence. The following result established the possibility of getting normal topologies with club guessing sequences even in the presence of Martin's Axiom.

Theorem 4 ([4])**.** *If ZFC is consistent, then it is also consistent with ZFC that $\mathsf{MA}+\neg\mathsf{CH}$ holds and there exists a strong club guessing and a 2-stationary hitting guessing sequence $\overrightarrow{E}$. Moreover, the resulting topology τ is normal.*

However, the guessing sequence given by this theorem is not built around a required partition of ω_1 into stationary sets. The difficulty in obtaining a Dowker topology using a guessing sequence is guaranteeing not only that the $E_\alpha \subseteq \bigcup_{k=0}^{n} S_k$ but also that the τ-closure of E_α stays inside the first n-levels of the space, $\bigcup_{k=0}^{n} S_k$. This we were not able to obtain.

3. Construction of the guessing sequence

We construct a ladder system $\overrightarrow{E} = \langle E_\gamma : \gamma \in \mathrm{Lim}\,(\omega_1)\rangle$ with enough properties. The method to construct this ladder system is different from the one used in [5]. There, a forcing iteration of non-proper posets was used; here, we use Gödel's axiom of constructibility, $V = L$.

Proposition 5. *Assume $V = L$. If $\{S_n : n \in \omega\}$ is a partition of ω_1 into stationary subsets, then there is a strong club guessing ladder system $\overrightarrow{E}$ such that $\alpha \in S_n \Rightarrow E_\alpha \subseteq \bigcup_{k=0}^{n} S_k$, and moreover, $\overrightarrow{E}$ has the following property: If $A, B \subseteq \bigcup_{k=0}^{n} S_k$ are stationary, then there exists a stationary $T \subseteq S_{n+1}$ such that*

$$\gamma \in T \Rightarrow \sup\,(A \cap E_\gamma) = \gamma = \sup(B \cap E_\gamma).$$

Proof: Fix the partition $\{S_n : n \in \omega\}$ of ω_1 into stationary sets. To aid in the notation, let $f : \omega_1 \to \omega$ be such that $S_n = f^{-1}(\{n\})$ for each $n \in \omega$. For each limit ordinal $\gamma \in \mathrm{Lim}\,(\omega_1)$, let $\mathcal{A}_\gamma$ be defined by $\alpha \in \mathcal{A}_\gamma$ if and only if

(i) $L_\alpha \vDash \mathsf{ZF}^-$ (i.e., ZF without the power set axiom),
(ii) $\gamma = \omega_1^{L_\alpha}$, and
(iii) $f \restriction \gamma \in L_\alpha$ and $L_\alpha \vDash$ "$f \restriction \gamma$ codes a partition into stationary sets."

Then $|\mathcal{A}_\gamma| \le \aleph_0$ for each $\gamma \in \mathrm{Lim}\,(\omega_1)$, since $\{\rho \in \omega_1 : L_\rho \prec L_{\omega_1}\}$ is unbounded in ω_1 and $L_{\omega_1} \vDash \gamma < \omega_1$. Note that $\{\gamma < \omega_1 : \mathcal{A}_\gamma \neq \emptyset\}$ contains a club set. Now let

$$\mathcal{G}_\gamma = \{C \subseteq \gamma : C \text{ is club in } \gamma \;\&\; (\exists \alpha \in \mathcal{A}_\gamma)\,(C \in L_\alpha)\}\,.$$

Then $\mathcal{G}_\gamma$ is countable and closed under finite intersections. Finally, let $\mathcal{H}_\gamma$ be defined as $\langle A^0, A^1\rangle \in \mathcal{H}_\gamma$ if and only if

(i) $(\forall i \in 2)\,(A^i \subseteq \gamma)$,
(ii) $(\exists \alpha \in \mathcal{A}_\gamma)\,(\forall i \in 2)\,(A^i \in L_\alpha)$,

(iii) $(\forall C \in \mathcal{G}_\gamma)\,(\forall i \in 2)\left(C \cap A^i \neq \emptyset\right)$,
(iv) $(\forall i \in 2)\left(\forall \xi \in A^i\right)(f(\xi) < f(\gamma))$.

Note that given $\left\langle A^0, A^1\right\rangle \in \mathcal{H}_\gamma$, we have that $L_\alpha \models$ "A^i is stationary in γ," for a suitable $\alpha \in \mathcal{A}_\gamma$. $\mathcal{H}_\gamma$ is also countable.

We will use induction to construct each term of the guessing ladder. For instance, the member E_γ will be constructed as follows. Let us start by assuming $\gamma \in S_m$ and enumerating $\mathcal{G}_\gamma$ as $\{C_n\}_{n\in\omega}$ and $\mathcal{H}_\gamma$ as $\left\{\left\langle A_n^0, A_n^1\right\rangle\right\}_{n\in\omega}$, where in the list of elements of $\mathcal{H}_\gamma$, we mention each element $\aleph_0$ times; and, in case $\mathcal{A}_\gamma$ has no maximum element, choose a cofinal sequence $\{\alpha_n : n \in \omega\}$ of $\mathcal{A}_\gamma$ so that for all $n \in \omega$, $\left\langle A_m^0, A_m^1\right\rangle, C_m \in L_{\alpha_n}$, for all $m \leq n$. If $\mathcal{A}_\gamma$ has maximum α_0, then let $\alpha_n = \alpha_0$ for all $n \in \omega$.

Choose an increasing sequence $\langle \delta_n : n \in \omega\rangle$ cofinal in γ. We know that for each $i \in 2$, $L_{\alpha_0} \models$ "$A_0^i \cap C_0$ is stationary in γ"; working inside L_{α_0}, we can choose $\gamma_0 \in A_0^0 \cap C_0 \setminus \delta_0$ and $\gamma_1 \in A_0^1 \cap C_0 \setminus \gamma_0$; then apply the same, now inside L_{α_1}, and so on. In general, we select $\gamma_{2n} \in A_n^0 \cap \bigcap_{m\leq n} C_m \setminus \max\{\delta_n, \gamma_{2n-1}\}$ and $\gamma_{2n+1} \in A_n^1 \cap \bigcap_{m\leq n} C_m \setminus \gamma_{2n}$. Finally, we let

$$E_\gamma = \{\gamma_n : n \in \omega\}.$$

Clearly, E_γ defined this way will be a cofinal in γ, and for each $\left\langle A^0, A^1\right\rangle \in \mathcal{H}_\gamma$, we have that $\sup\left(E_\gamma \cap A^i\right) = \gamma$, for $i \in 2$. Observe that if $\gamma \in S_m$, then $E_\gamma \subseteq \bigcup_{k=0}^m S_k$ as E_γ contains only points from the A_n^i forming the pairs elements of $\mathcal{H}_\gamma$.

Claim 1. If $C \subseteq \omega_1$ is a club and $\left\langle A^0, A^1\right\rangle$ is a pair of stationary subsets such that $f(A^0 \cup A^1) \subseteq m+1$, then there exist a stationary $T \subseteq S_{m+1}$ and club $K \subseteq \omega_1$ such that $\left\langle A^0 \cap \gamma, A^1 \cap \gamma\right\rangle \in \mathcal{H}_\gamma$ for all $\gamma \in T$, and $C \cap \gamma \in \mathcal{G}_\gamma$ for all $\gamma \in K$.

Proof of Claim 1: Let $\left\langle A^0, A^1\right\rangle$ and C be given as above; we will find T and K. By recursion, define a sequence of elementary submodels $M_\nu \prec L_{\omega_2}$, $\nu < \omega_1$, as follows:

: M_0 is the smallest $M \prec L_{\omega_2}$ such that $\left\langle A^0, A^1\right\rangle, C \in M$,
: $M_{\nu+1}$ is the smallest $M \prec L_{\omega_2}$ such that $M_\nu \cup \{M_\nu\} \subseteq M$,
: $M_\xi = \bigcup_{\nu<\xi} M_\nu$ if ξ is a limit ordinal.

Since we are assuming $V = L$ and $M_\nu \prec L_{\omega_2}$, it follows that $M_\nu \cap L_{\omega_1}$ is transitive. Let $\alpha_\nu = M_\nu \cap \omega_1$. Then $\langle \alpha_\nu : \nu < \omega_1\rangle$ is an increasing continuous sequence in ω_1. Let

$$\pi_\nu : M_\nu \cong L_{\beta_\nu}.$$

Clearly, then,

$$\pi_\nu \restriction L_{\alpha_\nu} = id \restriction L_{\alpha_\nu}, \quad \pi_\nu(\omega_1) = \alpha_\nu,$$

$$\pi_\nu(C) = C \cap \alpha_\nu, \quad \pi_\nu(\langle A^0, A^1 \rangle) = \langle A^0 \cap \alpha_\nu, A^1 \cap \alpha_\nu \rangle.$$

Consider K the set of all limit points of the set $\{\alpha_\nu : \nu < \omega_1\}$. Then K is club in ω_1. Let $\gamma \in K$ be given. For some limit ordinal $\lambda < \omega_1$,

$$\gamma = \sup_{\nu<\lambda} \alpha_\nu = \sup_{\nu<\lambda} \beta_\nu$$

and hence, $\gamma = \alpha_\lambda$. To see this, it suffices to prove that for all $\nu < \omega_1$, $\alpha_\nu < \beta_\nu < \alpha_{\nu+1}$. Clearly $\alpha_\nu < \beta_\nu$. But β_ν is definable from M_ν since L_{β_ν} is the transitive collapse of M_ν, and moreover, this definition relativises to L_{ω_2}. So as $M_\nu \in M_{\nu+1} \prec L_{\omega_2}$, we have $\beta_\nu \in M_{\nu+1}$. Hence, $\beta_\nu \in \alpha_{\nu+1}$, and the affirmation is confirmed.

Note that $\beta_\lambda \in \mathcal{A}_\gamma$, since we have that $L_{\beta_\lambda} \models$ "$\gamma = \omega_1$" and $L_{\beta_\lambda} \models \mathsf{ZF}^-$; hence, $C \cap \gamma = \pi_\lambda(C) \in L_{\beta_\lambda}$ and, of course, $L_{\beta_\lambda} \models \pi_\lambda(C)$ is club set in γ. Thus, $C \cap \gamma \in \mathcal{G}_\gamma$.

Also, $\langle A^0 \cap \gamma, A^1 \cap \gamma \rangle \in L_{\beta_\lambda}$. To obtain the rest of our Claim 1, we need to find the stationary set T. For this, let $\mu_\gamma = \sup \mathcal{A}_\gamma$ for each $\gamma < \omega_1$ such that $\mathcal{A}_\gamma \neq \emptyset$. Let us define

$$E = \left\{ \gamma \in S_{m+1} : (\forall i \in 2)\, (L_{\mu_\gamma} \models \text{"}A^i \cap \gamma \text{ is stationary in } \gamma = \omega_1^{L_{\mu_\gamma}}\text{"}) \right\}.$$

Claim 2. The set E is stationary in ω_1.

Proof of Claim 2: Let F be a club subset of ω_1. To find an ordinal in the intersection of F and E, let $\xi < \omega_2$ be the least ordinal such that $\{S_k\}_{k\in\omega}, F, A^0, A^1 \in L_\xi$, and for $n < \omega$, let κ_n be the $(n+1)$-th ordinal greater than ξ such that $L_{\kappa_n} \models \mathsf{ZF}^-$.

For each $n \in \omega$, define countable submodels $M_\nu^n \prec L_{\kappa_n}$ for $\nu < \omega_1$ as follows:

- : M_0^n is the smallest $M \prec L_{\kappa_n}$ such that $\{S_k\}_{k\in\omega}, F, A^0, A^1 \in M$,
- : $M_{\nu+1}^n$ is the smallest $M \prec L_{\kappa_n}$ such that $M_\nu^n \cup \{M_\nu^n\} \subseteq M$, and
- : $M_\nu^n = \bigcup_{\iota<\nu} M_\iota^n$, when ν is a limit ordinal.

Now, set $\alpha^n_\nu = M^n_\nu \cap \omega_1$. Then the sequence $\langle \alpha^n_\nu : \nu < \omega_1 \rangle$ is normal for every $n \in \omega$. Put $G_n = \{\nu < \omega_1 : \nu = \alpha^n_\nu\}$ and let $G = \bigcap \{G_n : n \in \omega\}$. Then

$$\nu \in G \Rightarrow \nu = \alpha^n_\nu,$$

for all $n \in \omega$.

Fix $\nu = \min(G \cap S_{m+1})$, and consider transitive collapsing maps $\pi_n : M^n_\nu \cong L_{\gamma^n_\nu}$ for $n \in \omega$. Then $\pi_n \restriction \nu = id \restriction \nu$, $\pi_n(\omega_1) = \nu$, $\pi_n(S_k) = S_k \cap \nu$ for all $k \in \omega$, $\pi_n(F) = F \cap \nu$, $\pi_n(A^i) = A^i \cap \nu$, and $\gamma^n_\nu \in \mathcal{A}_\nu$. Let $\gamma_\nu = \sup_{n<\omega} \gamma^n_\nu$. Then $L_{\gamma_\nu} \models$ "$\nu = \omega_1$," $L_{\gamma_\nu} \models$ "$S_k \cap \nu$ is stationary" for every $k \in \omega$, $L_{\gamma_\nu} \models$ "$A^i \cap \nu$ is stationary" for every $i \in 2$, and $L_{\gamma_\nu} \models$ "$F \cap \nu$ is cofinal in ν."

For every $\mu \geq \gamma_\nu$, we want to prove $\mu \notin \mathcal{A}_\nu$ to have that $\mu_\nu = \gamma_\nu$. Suppose $L_\mu \models$ "$\nu = \omega_1$." Let $\overline{\xi}_\nu$ be the least ordinal such that

$$\{S_k \cap \nu\}_{k\in\omega}, \ \{A^i \cap \nu\}_{i\in 2}, \ F \cap \nu \in L_{\overline{\xi}_\nu}.$$

Then we have $\pi_n(\xi) = \overline{\xi}_\nu$, for all $n \in \omega$. It follows that for $n > 0$,

$$(\pi_n)^{-1}(\gamma^0_\nu) = \kappa_0, \ (\pi_n)^{-1}(\gamma^1_\nu) = \kappa_1, \ldots, (\pi_n)^{-1}(\gamma^{n-1}_\nu) = \kappa_{n-1},$$

and γ^n_ν is the $(n+1)$-th ordinal greater than $\overline{\xi}_\nu$ such that $L_{\gamma^n_\nu} \models \mathsf{ZF}^-$. Thus, $\langle \gamma^n_\nu : n \in \omega \rangle$ is definable in L_{γ_ν}, and hence, L_{γ_ν} cannot be a model of ZF^-, so $\gamma_\nu \notin \mathcal{A}_\nu$. If $\mu > \gamma_\nu$, then $\langle \gamma^n_\nu : n \in \omega \rangle \in L_\mu$. Working inside L_μ, we can define the L_μ-versions, $\overline{G}_n$, of the club's $G_n \subseteq \omega_1$ defined earlier (with $\overline{\xi}_\nu$ in place of ξ, γ^n_ν in place of κ_n, etc.). Then $\overline{G}_n = G_n \cap \nu$ for all $n \in \omega$. Thus,

$$L_\mu \models \text{“}\bigcap_{n\in\omega} G_n \cap \nu \text{ is club in } \nu.\text{”}$$

Hence, if $\mu \in \mathcal{A}_\gamma$, then $L_\mu \models \left(\bigcap_{n\in\omega} G_n \cap \nu\right) \cap (S_{m+1} \cap \nu) \neq \emptyset$. Nevertheless, this is impossible since $\nu = \min(G \cap S_{m+1})$. Therefore, μ is not an element of $\mathcal{A}_\gamma$, and hence, $\mu_\nu = \gamma_\nu$. In summary,

$$\begin{aligned} L_{\gamma_\nu} &\models \text{“}\nu = \omega_1,\text{”} \\ L_{\gamma_\nu} &\models \text{“}A^i \cap \nu \text{ is stationary,” for every } i \in 2, \text{ and} \\ L_{\gamma_\nu} &\models \text{“}F \cap \nu \text{ is cofinal in } \nu.\text{”} \end{aligned}$$

The first two parts, together with the choice of ν, imply that $\nu \in E$, and the third one implies that $\nu \in F$ since F is closed. This completes the proof of Claim 2.

So, the set E is stationary and put $T = E \cap K$. For if $\gamma \in T$, let λ so that $\gamma = \alpha_\lambda$. As we have seen, $\beta_\lambda \in \mathcal{A}_\gamma$, $L_{\mu_\gamma} \models$ "$\gamma = \omega_1$," and $L_{\mu_\gamma} \models$ "$A^i \cap \gamma$ is stationary in γ" for every $i \in 2$, although L_{μ_γ} is not a model of ZF^-. However, if $C \in L_\beta$ is a club subset of γ for some $\beta \in \mathcal{A}_\gamma$, then $C \in L_{\mu_\gamma}$ and C is still a club since this notion is absolute. Therefore, for all $i \in 2$, $L_{\mu_\gamma} \models C \cap (A^i \cap \gamma) \neq \emptyset$ as we needed to prove. So, $\left(A^0 \cap \gamma, A^1 \cap \gamma\right) \in \mathcal{H}_\gamma$.

And this completes the proof of the proposition. □

Take a ladder system $\overrightarrow{E}$ given by Proposition 5. Applying Proposition 2 and considering the topology τ associated to $\overrightarrow{E}$, we obtain a Dowker topology on ω_1.

4. A LOCALLY COMPACT FIRST COUNTABLE MODIFICATION OF THE SPACE

The space in the previous section is locally countable, but it is neither first countable nor locally compact. We want to present a modification of the construction obtaining a locally compact example. In particular, we prove the following theorem.

Theorem 6. *Assume $V = L$ and let τ be the Dowker topology obtained from the ladder system $\overrightarrow{E}$ constructed in the previous section. There is another topology ρ on ω_1 that is finer than the order topology and coarser than τ such that (ω_1, ρ) is first countable, locally compact, and a Dowker space.*

Proof: Let $\overrightarrow{E}$ be the ladder system constructed in the previous section. Instead of using final segments of its elements as weak neighborhoods, we will choose suitable compact neighborhoods for the points using the ladder system $\overrightarrow{E}$ in the usual way (e.g., see [2]). The resulting topology will be coarser than the topology τ, but neighborhoods will still "look to the left." Hence, the resulting topology will again fail to be countably paracompact. To guarantee that normality will not be destroyed, we need to build separations for possible disjoint closed subsets. For this we need to assume $\diamondsuit^+$.

We are going to use a $\diamondsuit^+$ sequence that captures quintuples of subsets of ω_1. That is, we will assume

$\diamondsuit^+$ there is a sequence $\langle \mathcal{D}_\alpha : \alpha \in \mathrm{Lim}\,(\omega_1)\rangle$ so that for any subsets A, B, C, and K of ω_1, we have that there is a club

$D \subseteq \omega_1$ such that

$$(\forall \gamma \in D)\,(\langle A\cap\gamma, B\cap\gamma, C\cap\gamma, K\cap\gamma, D\cap\gamma\rangle \in \mathcal{D}_\gamma)\,.$$

For a quintuplet $q = \langle A, B, C, K, D\rangle$, we are going to denote by $q \restriction \gamma$ the quintuplet $\langle A\cap\gamma, B\cap\gamma, C\cap\gamma, K\cap\gamma, D\cap\gamma\rangle$. We are also going to use club guessing sequence $\{E_\alpha : \alpha \in \mathrm{Lim}\,(\omega_1)\}$ constructed earlier.

We will define, by recursion on $\beta \in \omega_1$, topologies ρ_β on $[0, \beta)$. Having defined ρ_β, we will say that a quintuplet $q = \langle A, B, C, K, D\rangle \in \mathcal{D}_\beta$ is *important* if

(1) A and B are ρ_β-closed,
(2) $K \subseteq C$ and both are closed in β,
(3) $E_\alpha \subseteq^* C$ for all $\alpha \in K$,
(4) $(A \cup C) \cap B = \emptyset$,
(5) D is closed and for all $\alpha \in D$, $q \restriction \alpha \in \mathcal{D}_\alpha$.

Notice that if $q = \langle A, B, C, K, D\rangle \in \mathcal{D}_\beta$ is important, then so is $q \restriction \alpha$ for every $\alpha \in D$.

We then also define ρ_β-open sets U^0_q and U^1_q for all important quintuplets $q \in \mathcal{D}_\beta$. The topologies and pairs of open sets satisfy the following inductive hypotheses:

(1) For each $\gamma < \beta$, ρ_β is a "conservative extension" of ρ_γ. That is, the ρ_β subspace topology on $[0, \gamma)$ and ρ_γ coincide.
(2) ρ_β is a T_1, zero-dimensional locally compact topology on $[0, \beta)$ that is finer than the order topology and coarser than the τ subspace topology where τ is the topology associated with the ladder system $\overrightarrow{E}$.
(3) If β is a limit ordinal and $q = \langle A, B, C, K, D\rangle \in \mathcal{D}_\beta$ is important, then
   (a) U^0_q and U^1_q form an open separation for $A \cup C$ and B in β with the topology ρ_β,
   (b) these open separations are coherent along $K \cap D$; that is, $U^i_{q\restriction\gamma} = U^i_q \cap \gamma$, for each $i \in 2$ and each $\gamma \in K \cap D$, and
   (c) if $E_\beta \subseteq^* C$, then $U^0_q \cup \{\beta\}$ is an open subset of $[0, \beta]$ in the topology $\rho_{\beta+1}$.

We assume we have topologies ρ_β defined on $[0, \beta)$ for all $\beta < \alpha$, satisfying the above inductive hypotheses.

Now, we consider the case where α is a limit. We need to define the topology ρ_α and all the separations U^i_q for all important quintuplets $q \in \mathcal{D}_\alpha$, then define $\rho_{\alpha+1}$ so that (3)(c) is satisfied.

First, since α is a limit, by (1), we must define $\rho_\alpha = \bigcup_{\beta<\alpha} \rho_\beta$. It follows from the definition that (1) and (2) are satisfied.

Now let $q = \langle A, B, C, K, D\rangle \in \mathcal{D}_\alpha$ be important.

Case 1: If $K \cap D$ is unbounded in α, by coherence, i.e., (3)(b), we must define $U^i_q = \bigcup_{\beta \in K\cap D} U^i_{q\restriction\beta}$ for each $i \in 2$.

Case 2: If $K \cap D$ is bounded in α, let $\gamma = \max K \cap D < \alpha$. Then $U^0_{q\restriction\gamma}$ and $U^1_{q\restriction\gamma}$ form an open separation of $(A \cup C) \cap \gamma$ and $B \cap \gamma$. Since $\gamma \in K$, we have that $\gamma \in C$ and $E_\gamma \subseteq^* C$, so it follows by (3)(c) that $U^0_{q\restriction\gamma} \cup \{\gamma\}$ is open. Since every countable regular space is normal, we also know that there are ρ_α-open sets W_0 and W_1 which are an open separation for $(A \cup C) \cap (\gamma, \alpha)$ and $B \cap (\gamma, \alpha)$. Then we let

$$U^0_q = U^0_{q\restriction\gamma} \cup \{\gamma\} \cup W_0$$

and

$$U^1_q = U^1_{q\restriction\gamma} \cup W_1.$$

It follows directly from the construction that the open separations satisfy (3)(a) and (3)(b). So we must now define the topology $\rho_{\alpha+1}$, taking care of (1), (2), and (3)(c). Enumerate as

$$\{q_n = \langle A_n, B_n, C_n, K_n, D_n\rangle : n \in \omega\}$$

all important quintuplets $q \in \mathcal{D}_\alpha$ with the property that $E_\alpha \subseteq^* C$. Let $E_\alpha = \{\beta_m : m \in \omega\}$. Fix an increasing sequence $\{k_n : n \in \omega\} \subseteq \omega$ such that $\beta_m \in C_n$ for all n and all $m > k_n$.

Finally, we define compact open neighborhoods around α in the standard way as follows. First, ignore the first k_0 elements of E_α. Since $\{\beta_i : k_0 < i \leq k_1\} \subseteq U^0_{q_0}$, we may fix a compact neighborhood $W_0 \subseteq [0, \beta_{k_1}]$ of the set $\{\beta_i : k_0 < i \leq k_1\}$ that lies inside $U^0_{q_0}$. In general, for $n \geq 1$, we may fix a compact neighborhood W_n of $\{\beta_i : k_n < i \leq k_{n+1}\}$ that lies inside $\bigcap_{k\leq n} U^0_{q_k} \setminus (\beta_{k_n}, \beta_{k_{n+1}}]$. Then a neighborhood base at α is given by the sets

$$W(\alpha, k) = \{\alpha\} \cup \bigcup_{n\geq k} W_n, \quad k \in \omega.$$

We let $\rho_{\alpha+1}$ be the topology generated by taking $\rho_\alpha \cup \{W(\alpha,k) : k \in \omega\}$ as a basis. It is straightforward to verify that (1) and (2) are satisfied. And (3)(c) will follow since for all k, $W(\alpha,k) \subseteq U^0_{q_k} \cup \{\alpha\}$.

This completes the recursive construction for α a limit and for α the successor of a limit. In the case that $\alpha = \beta + 1$ and β is a successor, we have defined the topology ρ_β on $[0,\beta) = [0,\beta^-]$. We extend the topology to $\rho_{\beta+1}$ by declaring $\{\beta\}$ to be isolated. Then hypotheses (1) and (2) are easily satisfied. And since $\mathcal{D}_\alpha$ is only defined in the case that α is a limit, hypotheses (3)(a), (3)(b) and (3)(c) are vacuously satisfied, and we have completed the recursive construction.

Finally, let $\rho = \bigcup_{\alpha\in\omega_1} \rho_\alpha$. Since ρ is a coarser topology than τ, and the decreasing sequence of τ-closed sets witnessing the failure of countable paracompactness are also ρ-closed, the space is not countably paracompact. We are left to show that the coarser topology is still normal.

Lemma 7. *The space (ω_1, ρ) is normal.*

Proof: Let A_0 and A_1 be stationary subsets of ω_1. Then, since ρ is coarser than τ, there must exist $\gamma \in \mathrm{Lim}\,(\omega_1)$ such that $\gamma \in \mathrm{cl}_\rho(A_i)$ for each $i \in \{0,1\}$. Therefore, no two stationary sets have disjoint closures. Thus, if we consider two disjoint ρ-closed subsets $A \subseteq \omega_1$ and $B \subseteq \omega_1$, without loss of generality, we may assume that B is not stationary, and hence, there is a club $C \subseteq \omega_1$ such that $C \cap B = \emptyset$. For C, we know there is a club K such that $\gamma \in K \Rightarrow E_\gamma \subseteq^* C$ as $\overrightarrow{E}$ is a strong club guessing ladder system. Using our $\diamondsuit^+$-sequence, we know that there is a club $D \subseteq \omega_1$ such that $q \restriction \gamma \in \mathcal{D}_\gamma$ for $\gamma \in D$, where $q = \langle A, B, C, K, D\rangle$. And moreover, for $\gamma \in D \cap K$, we have built ρ-open subsets $U^0_{q\restriction\gamma}$ and $U^1_{q\restriction\gamma}$ so that

(1) $(A \cup C) \cap [0,\gamma) \subseteq U^0_{q\restriction\gamma} \subseteq [0,\gamma]$,
(2) $B \cap [0,\gamma) \subseteq U^1_{q\restriction\gamma} \subseteq [0,\gamma]$,
(3) $\xi < \eta \quad \& \quad \xi \in D \cap K \cap \eta \Rightarrow U^i_{q\restriction\xi} = U^i_{q\restriction\eta} \cap [0,\xi]$, and
(4) $U^0_{q\restriction\gamma} \cap U^1_{q\restriction\gamma} = \emptyset$.

Let $U^0_q = \bigcup \left\{ U^0_{q\restriction\xi} : \xi \in D \cap K \right\}$ and $U^1_q = \bigcup \left\{ U^1_{q\restriction\xi} : \xi \in D \cap K \right\}$ to obtain a separation for A and B.

And this completes the proof of Theorem 6. □

References

[1] U. Abraham and S. Todorcevic, *Martin's axiom and first-countable S- and L-spaces*, in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 327–346

[2] Peter de Caux, *A collectionwise normal weakly θ-refinable Dowker space which is neither irreducible nor realcompact*, Topology Proc. **1** (1976), 67–77 (1977).

[3] C. H. Dowker, *On countably paracompact spaces*, Canadian J. Math. **3** (1951), 219–224.

[4] Fernando Hernández-Hernández, *Guessing sequences and topologies on ω_1*. Ph.D. Dissertation. York University, 2004.

[5] Fernando Hernández-Hernández and Tetsuya Ishiu, *A perfectly normal nonrealcompact space consistent with $\mathsf{MA}_{\aleph_1}$*, Topology Appl. **143** (2004), no. 1-3, 175–188.

[6] Paul Larson and Stevo Todorcevic, *Katětov's problem*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1783–1791.

[7] Paul J. Szeptycki, *Small Dowker spaces*, in Open Problems in Topology II. Ed. Elliott Pearl. Amsterdam, Boston: Elsevier, 2007. 233–239

[8] Teruyuki Yorioka, *Rudin's Dowker space in the extension with a Suslin tree*, Fund. Math. **201** (2008), no. 1, 53–89.

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