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A perfectly normal nonrealcompact space consistent with $\mathsf{MA}_{\aleph_1}$

Fernando Hernández-Hernández [a,*,1], Tetsuya Ishiu [b]

[a] *York University, Toronto, Ont. M3J1L3, Canada*
[b] *University of California, Irvine, CA 92697-3875, USA*

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Abstract

We settle a conjecture due to R.L. Blair by proving that it is consistent with Martin's Axiom to have a perfectly normal nonrealcompact space of cardinality $\aleph_1$.

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1. Introduction

Realcompact spaces were defined and investigated by Hewitt and Nachbin. Hewitt demonstrated the importance of these spaces by proving the isomorphism theorem: "If X and Y are realcompact spaces, then $C(X)$ is isomorphic to $C(Y)$ if and only if X is homeomorphic to Y". He also derived many of the properties of realcompact spaces, often shared in common with those enjoyed by compact spaces. There are several characterizations of realcompact spaces, we will use: a Tychonoff space X is *realcompact* if every z-ultrafilter on X with the countable intersection property has non-empty total intersection. A z-ultrafilter is a filter consisting of zero-sets, i.e., sets of the form $f^{-1}(0)$, for some real-valued continuous function f, which is maximal among the z-filters. A comprehensive study of realcompact spaces is done in [10,20].

* Corresponding author.
E-mail addresses: fhernand@mathstat.yorku.ca (F. Hernández-Hernández), tishiu@math.uci.edu (T. Ishiu).
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On the other hand, perfectly normal spaces form a wide class of topological spaces; in particular, any metric space is a perfectly normal space. Perfect normality is a property closely related to metrization properties and it usually plays an important role in many different proofs. A space X is said *perfectly normal* if it is normal and every closed subset of X is a G_δ (equivalently: every closed subset is a zero-set). In [4] Blair introduced the class of weakly perfectly normal spaces to study z-embedded subsets. A subset S is *z-embedded* in X in case each zero-set of S is the restriction to S of a zero set of X. A space X is *weakly perfectly normal* if every subset of X is z-embedded in X. Every perfectly normal space is weakly perfectly normal and any weakly perfectly normal space is completely normal. Blair asked whether there exists a perfectly normal space of cardinality less than the first measurable that is not realcompact. Blair proposed it as an open problem as early as 1962. It appears credited to him and Stephenson Jr in [11]. It also appears as a problem about a consequence $\mathsf{MA} + \neg\mathsf{CH}$ in [4], and more recently, it appears in [5] and later in [18] in the form

Problem 1. Is there a ZFC example of a perfectly normal space that is not realcompact? Or does $\mathsf{MA} + \neg\mathsf{CH}$ imply that every such space of cardinality less than the first measurable is realcompact?

Blair conjectured an affirmative answer to the last question. Katětov in [14] had showed that a paracompact space X is realcompact if and only if X does not have discrete subspaces of measurable cardinality. There are easy examples of countably paracompact spaces that are not realcompact. If a normal countably paracompact space has the property that every ultrafilter of closed subsets with the countable intersection property is fixed, then it is realcompact [15]. Perfect normality is a stronger property than countable paracompactness is, though paracompactness does not imply perfect normality. There are very few examples of perfectly normal nonrealcompact spaces (basically two until now). The discrete space of measurable cardinality and the Ostaszewski's classical construction [17] from $\diamondsuit$ are examples of perfectly normal nonrealcompact spaces. On the other direction, Weiss [21] showed that $\mathsf{MA} + \neg\mathsf{CH}$ implies that every perfectly normal countably compact space is compact. Blair and van Douwen showed [5] that under $\mathsf{MA} + \neg\mathsf{CH}$ every perfectly normal space X is *nearly realcompact*; that is, $\beta X \setminus \upsilon X$ is dense in $\beta X \setminus X$. Where υX and βX are the realcompactification and the Stone–Čech compactification of X, respectively. It is also easy to give realcompact (even compact) spaces that are not perfectly normal.

We settle in the negative Blair's conjecture that $\mathsf{MA} + \neg\mathsf{CH}$ implies that every perfectly normal space of cardinality less than the first measurable is realcompact. Swardson mentions that some partial results had been obtained; for instance, she proved in [19] that $\mathsf{MA} + \neg\mathsf{CH}$ implies that regular spaces of cardinality less than the first measurable in which closed sets have countable character are realcompact. Now we also know that normal spaces of small cardinality in which every subset is a G_δ are realcompact. These kinds of spaces are known as Q-set spaces; Balogh [2] showed they exists in ZFC.

Our topology is a mixture of the ideas behind the Ostaszewski's line and those behind the usual "ladder spaces" [16]. The main idea of our construction is to define a sequence of subsets of ω_1 which will serve as a weak neighbourhood base for a topology on ω_1 refining the usual order topology. Our first approach to the problem was by a certain combination of two of Nyikos' axioms. We knew the kind of guessing-meeting principle that would give us a perfectly normal space that is not realcompact and that we could have preserved after forcing with a ccc poset. Unfortunately that combination turned out to be inconsistent. Under $V = L$ we were able to produce [12] a guessing principle close to what we needed and we can construct a normal topology in which most closed sets are G_δ and the space is not realcompact, and this can be preserved by ccc posets of size $\aleph_1$. We think the principle that allows us to do this is of some interest on its own.

Here we use a quite new forcing notion to obtain a model with a guessing sequence. Further on, the forcing is made to take care of possible ccc forcing notions of small size over the generic model. This will allow us to preserve the important properties of our guessing sequence after a finite support iteration of ccc forcings to obtain a model of $\mathsf{MA} + \neg\mathsf{CH}$. The forcing we are using is a modification of one used by Foreman and Komjáth in [9]. See also [13].

2. Building the counterexample

Our terminology and notation follow the standards of contemporary set-theoretic topology. The few special symbols we use are next. For sets A and B of ordinal numbers we write $A \subseteq^* B$ if $A \setminus B$ is a bounded subset of $\sup A$; say it in another way, if there is some $\alpha < \sup A$ such that $(\forall \gamma > \alpha)(\gamma \in A \Longrightarrow \gamma \in B)$. For subsets A and B of ω_1, $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$. We often use interval notation; for example, $(\alpha, \beta]$ is the set of ordinals γ such that $\alpha < \gamma \leqslant \beta$. The set of all limit ordinals in ω_1 is denoted by $\mathrm{Lim}(\omega_1)$ and $[X]^\kappa$ denotes the family of all subsets of X with cardinality κ; $[X]^{<\kappa}$ and $[X]^{\leqslant\kappa}$ have the obvious meanings. "Club" means closed and unbounded set. We reserve the bar over a subset of ω_1 to denote its closure with respect to the order topology. Our forcing posets are downward directed. If Φ is a formula of the forcing language for $\mathbb{P}$, when we write $\mathbb{P} \Vdash \Phi$ we mean that the set $\{p \in \mathbb{P}\colon\ p \Vdash \Phi\}$ is dense in $\mathbb{P}$. We usually take the elements of the ground model as names for themselves but sometimes when a is an element of the ground model we indicate this by writing $p \Vdash \Phi(\check{a})$. Names are denoted by placing a dot over the object named. If $\mathbb{P}$ is a forcing notion over a model V, and G is a V-generic filter in $\mathbb{P}$, then we can have $\dot{\mathbb{R}}$ is a $\mathbb{P}$-name for a forcing notion over $V[G]$; in the extension $V[G]$ we may have an $\mathbb{R}$-name for a set of ordinals σ, then we denote by $\dot{\sigma}$ the $\mathbb{P}$-name for the $\dot{\mathbb{R}}$-name of the respective object in the generic extension of $V[G]$ by forcing with $\mathbb{R}$.

Definition 2. A sequence $\vec{E} = \langle E_\alpha\colon\ \alpha \in \mathrm{Lim}(\omega_1)\rangle$, where E_α is a cofinal subset of $\alpha \in \mathrm{Lim}(\omega_1)$ is called a guessing sequence for a family $\mathcal{A}$ if for each $X \in \mathcal{A}$ there is $\alpha \in \mathrm{Lim}(\omega_1)$ such that $E_\alpha \subseteq^* X$. If for each $X \in \mathcal{A}$ there are club many $\alpha \in \omega_1$ such that $E_\alpha \subseteq^* X$, then we say that the sequence $\vec{E}$ is strong guessing for the family $\mathcal{A}$. If $\mathcal{A}$ is the family $\mathcal{C}$ of all club subsets of ω_1 we simply say that the sequence $\vec{E}$ is club guessing or strong club guessing, respectively.

There is a topology $\tau(\vec{E})$ on ω_1 taking the elements of a guessing sequence $\vec{E}$ as weak neighbourhoods; that is, we define recursively neighbourhoods at each point of ω_1: for zero and successor ordinals α we stipulate $\{\alpha\}$ being open, and for limit ordinals γ, assuming we have defined neighbourhoods contained in $[0,\alpha]$ for each point $\alpha \in [0,\gamma)$, we define the neighbourhoods of γ to be sets of the form

$$\{\gamma\} \cup \bigcup \{W_\xi \colon \xi \in E_\gamma \setminus \beta\}, \tag{1}$$

where $\beta < \gamma$ and W_ξ is a neighbourhood of $\xi \in E_\gamma \setminus \beta$ in the space $[0,\gamma)$. We call this topology $\tau(\vec{E})$ *the topology associated with* $\vec{E}$. A similar topology was considered in [8].

Lemma 3. *Let $\vec{E} = \langle E_\gamma \colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$ be a guessing sequence and τ be the topology associated with $\vec{E}$. Then X is τ-closed if and only if for every limit ordinal $\gamma \in \omega_1$ such that $E_\gamma \cap X$ is unbounded in γ, $\gamma \in X$.*

Proof. The necessity is clear. Suppose that for every limit ordinal $\gamma \in \omega_1$, if $E_\gamma \cap X$ is unbounded in γ, then $\gamma \in X$. We shall show that X is τ-closed. We inductively show that for each $\gamma \in \omega_1 \setminus X$, there is a τ-open neighbourhood of γ disjoint from X. If γ is successor or 0, since γ is isolated, $\{\gamma\}$ is τ-open and disjoint from X. Suppose that γ is limit and for every $\xi \in \gamma \setminus X$, there is a τ-open neighbourhood W_ξ of ξ disjoint from X. By the assumption, we have $E_\gamma \cap X$ is bounded in γ. Let $\zeta = \sup(E_\gamma \cap X) + 1$. Then for every $\xi \in E_\gamma \setminus \zeta$, we have $\xi \in \gamma \setminus X$. By the inductive hypothesis, there exists a τ-open neighbourhood W_ξ of ξ disjoint from X. Let $W = \{\gamma\} \cup \bigcup\{W_\xi \colon \xi \in E_\gamma \setminus \zeta\}$. Then W is a neighbourhood of γ disjoint from X. □

The basic open neighbourhoods of γ in the topology just described are subsets of the interval $[0,\gamma]$, it follows that for every $\beta \in \omega_1$, the interval $[0,\beta)$ is τ-open. Applying Lemma 3 we see that the intervals $[0,\alpha]$ are τ-closed. Thus open intervals (α,β) are τ-open and $\tau(\vec{E})$ is finer than the usual order topology. Hence, $\tau(\vec{E})$ is a T_1-topology.

Definition 4. Let $\vec{E} = \langle E_\gamma \colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$ be a guessing sequence and let $X \subseteq \omega_1$. We say that $\gamma \in \omega_1$ is nice for X with respect to $\vec{E}$ if and only if both

(i) $\gamma \in X \Longrightarrow E_\gamma \subseteq^* X$, and
(ii) $\gamma \notin X \Longrightarrow E_\gamma \cap X$ is bounded in γ.

If $\vec{E}$ is clear from the context, we may simply say that γ is nice for X.

Definition 5. Let $\vec{E} = \langle E_\gamma \colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$ be a guessing sequence and $\tau = \tau(\vec{E})$. $\vec{E}$ is called a strong τ-guessing sequence if for every τ-closed subset F of ω_1 there exists a club subset D of ω_1 such that every $\gamma \in D$ is nice for F.

If F is a $\tau(\vec{E})$-closed subset of ω_1, then $\gamma \in \mathrm{Lim}(\omega_1)$ is nice for F if and only if $\gamma \in F$ is equivalent to $E_\gamma \subseteq^* F$. Notice also that every strong τ-guessing sequence is a strong club guessing sequence. To see this, let $\vec{E} = \langle E_\gamma \colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$ be a strong τ-guessing sequence and let C be a club subset of ω_1. Since $\tau(\vec{E})$ is finer than the order topology, every club subset of ω_1 is $\tau(\vec{E})$-closed. In particular, C is $\tau(\vec{E})$-closed. Let D be a club subset of ω_1 consisting of nice points for C. Without loss of generality, $D \subseteq C$. Then for every $\gamma \in D$, since γ is nice for C and $\gamma \in C$, $E_\gamma \subseteq^* C$. Conversely, if $\vec{E}$ is strong club guessing and every stationary $\tau(\vec{E})$-closed subset contains a club subset of ω_1, then $\vec{E}$ is strong $\tau(\vec{E})$-guessing.

Lemma 6. *If $\vec{E} = \langle E_\alpha\colon \alpha \in \mathrm{Lim}(\omega_1)\rangle$ is a strong τ-guessing sequence and the sets E_α are closed in the order topology, then $\tau(\vec{E})$ is a perfectly normal zero-dimensional topology. Moreover, if any stationary $\tau(\vec{E})$-closed set contains a club subset of ω_1, then the club filter generates a countable complete z-ultrafilter, and hence $(\omega_1, \tau(\vec{E}))$ is not a realcompact space.*

Proof. The topology $\tau = \tau(\vec{E})$ is locally countable, thus if τ is normal it will be zero-dimensional. To prove normality we first establish that τ is regular by induction. Suppose we have that $([0,\alpha), \tau \restriction [0,\alpha))$ is regular for all $\alpha \leqslant \gamma$, where $\tau \restriction [0,\alpha]$ is the subspace topology on $[0,\alpha]$. We need to show that $([0,\gamma], \tau \restriction [0,\gamma])$ is regular as well. The unique non-trivial pair for which we must prove regularity is that in which we have a limit ordinal γ and a τ-closed subset F of $[0,\gamma]$ such that $\gamma \notin F$ and we want to exhibit disjoint neighbourhoods of F and γ, respectively. First, there is a neighbourhood W_γ of γ such that $F \cap W_\gamma = \emptyset$. We can assume that $W_\gamma = \{\gamma\} \cup \bigcup\{W_\xi\colon \xi \in E_\gamma \setminus \beta\}$. By the inductive hypothesis we can deduce that $[0,\gamma)$ is a normal space as it is countable and regular. If we prove that the closure, $\mathrm{cl}_\tau(E_\gamma \setminus \beta)$, of $E_\gamma \setminus \beta$ with respect to the new topology on $[0,\gamma]$ is disjoint from F, then using the normality of $[0,\gamma)$ we can choose disjoint τ-open sets U_0 and U_1 such that $F \subseteq U_0$ and $E_\gamma \setminus \beta \subseteq U_1$. Thus, taking $W'_\gamma = \{\gamma\} \cup \bigcup\{W_\xi \cap U_1\colon \xi \in E_\gamma \setminus \beta\}$ we would have $U_0 \cap W'_\gamma$ as we need it. And it is certainly the case that $F \cap \mathrm{cl}_\tau(E_\gamma \setminus \beta) = \emptyset$; for if $\alpha \in F \setminus \beta$, α cannot be a limit point of $E_\gamma \setminus \beta$ since otherwise, by our assumption that E_γ is closed under the order topology, α would be a member of $E_\gamma \setminus \beta$ contradicting that $W_\gamma \cap F = \emptyset$. Thus we can find a neighbourhood of α with no points from $E_\gamma \setminus \beta$, as we wanted to show. Since τ is a conservative extension of $\tau \restriction [0,\alpha]$, the regularity follows.

Let us now prove the normality. Let F and H be two disjoint τ-closed subsets of ω_1. Then by the assumption, there exists a club subset D of ω_1 such that every $\gamma \in D$ is nice for F and H. We shall define by induction on the elements of D two disjoint τ-open subsets U and W of ω_1 such that $F \subseteq U$ and $H \subseteq W$.

Let $\gamma_0 = \min(D)$. Since the subspace $[0,\gamma_0]$ is regular and countable, hence normal, there exist two disjoint τ-open subsets U_{γ_0} and W_{γ_0} such that $F \cap [0,\gamma_0] \subseteq U_{\gamma_0}$ and $H \cap [0,\gamma_0] \subseteq W_{\gamma_0}$. Define $U \cap [0,\gamma_0] = U_{\gamma_0}$ and $W \cap [0,\gamma_0] = W_{\gamma_0}$. Suppose now that we have defined $U \cap [0,\gamma]$ and $W \cap [0,\gamma]$ for some $\gamma \in D$. Let γ^+ be the next element of D, we need to define $U \cap (\gamma,\gamma^+]$ and $W \cap (\gamma,\gamma^+]$. Since the interval $(\gamma,\gamma^+]$ as a subspace is regular and countable, it is also normal. Note that $(\gamma,\gamma^+]$ is also clopen. So, we can find disjoint U_{γ^+} and W_{γ^+} τ-open subsets of $(\gamma,\gamma^+]$ such that $F \cap (\gamma,\gamma^+] \subseteq U_{\gamma^+}$ and $H \cap (\gamma,\gamma^+] \subseteq W_{\gamma^+}$. Then we define $U \cap [0,\gamma^+] = (U \cap [0,\gamma]) \cup U_{\gamma^+}$ and $W \cap [0,\gamma^+] = (W \cap [0,\gamma]) \cup W_{\gamma^+}$.

Finally suppose γ is a limit point of D and we have defined disjoint $U \cap [0,\xi]$ and $W \cap [0,\xi]$, for all $\xi \in D \cap \gamma$. Since $F \cap H = \emptyset$, we have three exclusive cases.

Case 1. $\gamma \notin F \cup H$. Then let $U \cap [0,\gamma] = \bigcup\{U \cap [0,\xi]\colon \xi \in D \cap \gamma\}$ and $W \cap [0,\gamma] = \bigcup\{W \cap [0,\xi]\colon \xi \in D \cap \gamma\}$.

Case 2. $\gamma \in F \setminus H$. Then let $U \cap [0,\gamma] = (U \cap [0,\gamma)) \cup \{\gamma\}$ and $W \cap [0,\gamma] = W \cap [0,\gamma)$. Clearly $U \cap [0,\gamma]$ and $W \cap [0,\gamma]$ are disjoint and $W \cap [0,\gamma]$ is τ-open. We claim that $U \cap [0,\gamma]$ is τ-open. Since $U \cap [0,\gamma)$ is τ-open, it suffices to show that there exists a neighbourhood N of γ such that $N \subseteq U \cap [0,\gamma]$. Since γ is nice for F, we have $E_\gamma \subseteq^* F$. Let $\beta < \gamma$ be such that $E_\gamma \setminus \beta \subseteq F$. Then for each $\xi \in E_\gamma \setminus \beta$, there is a neighbourhood $N_\xi \subseteq U \cap [0,\gamma)$. Let $N = \{\gamma\} \cup \bigcup\{N_\xi\colon \xi \in E_\gamma \setminus \beta\}$. Then N is a neighbourhood of γ contained in $U \cap [0,\gamma]$. Thus $U \cap [0,\gamma]$ is τ-open.

Case 3. $\gamma \in H \setminus F$. Then let $U \cap [0,\gamma] = U \cap [0,\gamma)$ and $W \cap [0,\gamma] = (W \cap [0,\gamma)) \cup \{\gamma\}$. By analogous argument as in the previous case they are τ-open and disjoint.

It is now trivial that U and W are τ-open and disjoint such that $F \subseteq U$ and $H \subseteq W$. Therefore (ω_1,τ) is normal.

Let us now prove that (ω_1,τ) is perfect. We shall prove that any τ-closed subset of (ω_1,τ) is a G_δ-set. If H is a τ-closed set, there is a club set $C \subseteq \omega_1$ such that for all $\gamma \in C$, γ is nice for H, which implies that $E_\gamma \subseteq^* H$ in case $\gamma \in H$. We can perform induction along C defining τ-open subsets $U_\xi(n)$, for $\xi \in C$, such that

(1) $H \cap [0,\xi] = \bigcap\{U_\xi(n)\colon n \in \omega\}$;
(2) $U_\xi(n) = U_\eta(n) \cap [0,\xi]$, whenever $\xi \leqslant \eta$.

For successor points of C there is no problem given that $H \cap (\xi,\xi^+]$ is G_δ since $(\xi,\xi^+]$ is T_1 and countable. So suppose ξ is a limit of ordinals in C and everything has been done accordingly so far. If $\xi \notin H$, then we only need to take the union of the sets previously defined. If $\xi \in H$, then a neighbourhood of ξ can be given by neighbourhoods of points from $E_\xi \cap H$: for each $n \in \omega$, we can find a neighbourhood $W_\xi(n) \subseteq U_n = \bigcup\{U_\lambda(n)\colon \lambda \in C \cap \xi\}$. So $U_\xi(n) = U_n \cup \{\xi\}$ will be τ-open containing $H \cap [0,\xi]$. Hence $H = \bigcap_{n\in\omega} U(n)$, where $U(n)$ is the union of all the $U_\xi(n)$, $\xi \in C$.

Lastly, let $\mathcal{F}$ be the family of all τ-closed subsets of ω_1 which contain a club subset of ω_1. Then $\mathcal{F}$ is a filter with the countable intersection property and $\bigcap\mathcal{F} = \emptyset$. The hypothesis easily implies that $\mathcal{F}$ is maximal. □

2.1. *The forcing construction*

Definition 7. Let $\mathbb{P}$ be a notion of forcing and let N be a countable elementary submodel of some $H(\lambda)$ with $\mathbb{P} \in N$.

(1) We say that a condition $q \in \mathbb{P}$ is totally $(N,\mathbb{P})$-generic if whenever D is a dense open subset of $\mathbb{P}$ that is in N, we can find a condition $p \in N \cap D$ with $q \leqslant p$. Said in another way, q is a lower bound for some N-generic filter $G \subseteq N \cap \mathbb{P}$.
(2) We say that $\mathbb{P}$ is totally proper if, given N as above, every $p \in N \cap \mathbb{P}$ has a totally $(N,\mathbb{P})$-generic extension $q \in \mathbb{P}$.

It is not difficult to prove that a notion of forcing is totally proper if and only if it is proper and the forcing adds no new reals. In the presence of properness, this is equivalent to the forcing adding no new ω-sequence of elements of the ground model. This was noticed by Eisworth and Roitman [7]; Definition 7 was introduced by them; although many people have used proper forcing adding no reals without naming it.

Proposition 8. *Suppose that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Then there exists a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there exists a guessing sequence $\vec{C}$ such that*

(i) *$\vec{C}$ is a strong club guessing sequence, and*
(ii) *if $\mathbb{R}$ is a ccc poset of size $\aleph_1$ and σ is an $\mathbb{R}$-name such that $\mathbb{R} \Vdash$ "σ is stationary and $\tau(\vec{C})$-closed", then $\mathbb{R} \Vdash$ "σ contains a club subset of ω_1".*

Proof. $\mathbb{P}$ will be an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leqslant \omega_2;\ \beta < \omega_2 \rangle$ of length ω_2 with countable support. We start our induction by adding a guessing sequence $\vec{C}$ with a forcing poset $\mathbb{Q}_0$ and we inductively construct $\dot{\mathbb{Q}}_\alpha$ for each $0 < \alpha < \omega_2$. In the course of induction, we shall also prove that

(1) (a) $\mathbb{P}_\alpha$ is $\aleph_2$-cc and totally proper,
(b) $\mathbb{P}_\alpha$ forces that $|\dot{\mathbb{Q}}_\alpha| \leqslant \aleph_1$,
(c) $\mathbb{P}_\alpha$ forces that $\vec{C}$ is a club guessing sequence, and
(d) $\mathbb{P}_\alpha$ forces that if $\dot{\mathbb{R}}$ is a ccc poset of size $\leqslant \aleph_1$ and $\dot{\sigma}$ is an $\dot{\mathbb{R}}$-name for a stationary $\tau(\vec{C})$-closed set, then

$$\bigl\{\gamma \in \mathrm{Lim}(\omega_1)\colon\ \dot{\mathbb{R}} \Vdash \sup\bigl(\dot{\sigma} \cap \dot{C}_\gamma\bigr) = \gamma\bigr\}$$

is stationary.

Note that in (1d), without loss of generality, we may assume that the domain of $\mathbb{R}$ is a subset of ω_1.

Define $\mathbb{Q}_0$ by: $p \in \mathbb{Q}_0$ if and only if p is a function such that $\mathrm{dom}(p)$ is a countable subset of $\mathrm{Lim}(\omega_1)$ and $p(\gamma)$ is a closed unbounded subset of γ, for all $\gamma \in \mathrm{dom}(p)$. If $G \subseteq \mathbb{Q}_0$ is generic, then for every $p, q \in G$ and $\gamma \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$, $p(\gamma) = q(\gamma)$. Let $C_\gamma = p(\gamma)$ for some (all) $p \in G$ with $\gamma \in \mathrm{dom}(p)$. It is easy to see that for every $\gamma \in \mathrm{Lim}(\omega_1)$, there exists a $p \in G$ with $\gamma \in \mathrm{dom}(p)$. Thus we get a guessing sequence $\vec{C} = \langle C_\gamma \colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$. Let $\dot{C}_\gamma$ be a name for C_γ, for each $\gamma \in \mathrm{Lim}(\omega_1)$.

By (1a) and (1b) of the inductive hypotheses, it is easy to see that for every $\alpha < \omega_2$, $\mathbb{P}_\alpha$ forces $2^{\aleph_1} = \aleph_2$. Thus, by a routine book-keeping argument, it is possible to consider sequences $\langle \dot{F}_\alpha \colon 0 < \alpha < \omega_2$ and α is even$\rangle$ of canonical names for club subsets of ω_1 which appear in $V^{\mathbb{P}_\beta}$ for some $\beta < \omega_2$, and $\langle (\dot{\mathbb{R}}_\alpha, \dot{\sigma}_\alpha) \colon \alpha < \omega_2$ and α is odd$\rangle$ of canonical names for pairs $(\mathbb{R}, \sigma)$ which appear in $V^{\mathbb{P}_\beta}$ for some $\beta < \omega_2$ such that $\mathbb{R}$ is a poset whose domain is a subset of ω_1 and σ is an $\mathbb{R}$-name for a subset of ω_1. Moreover, we can arrange our book-keeping so that every club and every such pair in the final extension occurs as some term of one of our sequences. When we work in the extension, we denote the evaluation of $\dot{F}_\alpha$, $\dot{\mathbb{R}}_\alpha$ and $\dot{\sigma}_\alpha$ by F_α, $\mathbb{R}_\alpha$ and σ_α, respectively.

First we shall explain how to define $\dot{\mathbb{Q}}_\alpha$ for $\alpha > 0$ assuming we have defined $\dot{\mathbb{Q}}_\beta$ for all $\beta < \alpha$. Let $G_\alpha \subseteq \mathbb{P}_\alpha$ be generic and we shall define a poset $\mathbb{Q}_\alpha$ in $V[G_\alpha]$. We let $\dot{\mathbb{Q}}_\alpha$ be a name for $\mathbb{Q}_\alpha$.

Case α is even. Let $\mathbb{Q}_\alpha$ be the standard poset shooting a club through $\{\gamma \in \mathrm{Lim}(\omega_1)\colon C_\gamma \subseteq^* F_\alpha\}$; see [1] for a precise definition of it. By (1c) of the inductive hypothesis, this definition makes sense.

Case α is odd. If $\mathbb{R}_\alpha$ is not a ccc poset or $\mathbb{R}_\alpha$ does not force that σ_α is stationary, then $\mathbb{Q}_\alpha$ is the trivial poset. Otherwise, let $\mathbb{Q}_\alpha$ be the standard poset shooting a club through $\{\gamma \in \mathrm{Lim}(\omega_1)\colon \mathbb{R}_\alpha \Vdash \sup(C_\gamma \cap \sigma_\alpha) = \gamma\}$. By (1d) of the inductive hypothesis, it makes sense.

In either case, if $\mathbb{Q}_\alpha$ is not trivial, let D_α be the club subset of ω_1 added at the αth stage and $\dot{D}_\alpha$ its $\dot{\mathbb{Q}}_\alpha$-name. If $\mathbb{Q}_\alpha$ is trivial, let $D_\alpha = \omega_1$ and $\dot{D}_\alpha = \check{\omega}_1$. Note that for every even $\beta > 0$, $D_\beta \subseteq F_\beta$ and for every odd β, $\mathbb{R}_\beta \Vdash D_\beta \subseteq \sigma_\beta$.

To see that we can carry out this induction, we need the following lemma.

Lemma 9. *Suppose that* (1a)–(1d) *are true for all $\beta < \alpha \leqslant \omega_2$. Let $p \in \mathbb{P}_\alpha$ and let $\dot{F}$, $\dot{\mathbb{R}}$, and $\dot{\sigma}$ be a $\dot{\mathbb{P}}_\alpha$-names such that $\mathbb{P}_\alpha$ forces that*

(a) *$\dot{F}$ is a club subset of ω_1,*
(b) *$\dot{\mathbb{R}}$ is a ccc poset,*
(c) *$\dot{\sigma}$ is a $\dot{\mathbb{R}}$-name such that $\dot{\mathbb{R}} \Vdash$ "$\dot{\sigma}$ is a stationary $\tau(\overrightarrow{C})$-closed subset of ω_1".*

Let M be a countable elementary submodel of some $H(\theta)$, for some large enough regular cardinal θ, such that $\{\mathbb{P}_\alpha, p, \overrightarrow{C}, \langle \dot{D}_\beta\colon \beta < \alpha\rangle, \dot{F}, \dot{\mathbb{R}}, \dot{\sigma}\} \in M$. Let $\delta = M \cap \omega_1$. Then there exists a $q \leqslant p$ such that

(i) *q is totally $(M, \mathbb{P}_\alpha)$-generic,*
(ii) *$q \Vdash \dot{C}_\delta \subseteq^* \dot{F}$, and*
(iii) *$q \Vdash$ "$\dot{\mathbb{R}} \Vdash \sup(\dot{C}_\delta \cap \dot{\sigma}) = \delta$".*

Suppose that the lemma and the inductive hypotheses are true for all $\beta < \alpha \leqslant \omega_2$. We shall show the inductive hypotheses for α. By a standard argument, if $\alpha < \omega_2$, then we can prove that $\mathbb{P}_\alpha$ has a dense subset of size $\leqslant \aleph_1$ and hence it is $\aleph_2$-cc. If $\alpha = \omega_2$, by a Δ-system lemma argument, $\mathbb{P}_{\omega_2}$ is $\aleph_2$-cc. By the previous lemma, $\mathbb{P}_\alpha$ is totally proper and hence it adds no new countable sequence of ordinals.

To see (1c), let $p \in \mathbb{P}_\alpha$ be arbitrary and $\dot{F}$ be a $\mathbb{P}_\alpha$-name such that $p \Vdash$ "$\dot{F}$ is a club subset of ω_1". Let $\dot{\mathbb{R}}$ be a $\mathbb{P}_\alpha$-name for the trivial poset and $\dot{\sigma}$ a $\mathbb{P}_\alpha$-name for a $\dot{\mathbb{R}}$-name for ω_1. Let M be a countable elementary submodel of $H(\theta)$, for some large enough regular cardinal θ as in the statement of Lemma 9 and set $\delta = M \cap \omega_1$. Let $q \leqslant p$ be as in the conclusion of Lemma 9. Then $q \Vdash \dot{C}_\delta \subseteq^* \dot{F}$.

For (1d), let $p \in \mathbb{P}_\alpha$ be arbitrary and let $\dot{D}$ be a $\mathbb{P}_\alpha$-name for a club subset of ω_1. Also let $\dot{\mathbb{R}}$ and $\dot{\sigma}$ be $\mathbb{P}_\alpha$-names such that $p \Vdash$ "$\dot{\mathbb{R}}$ is a ccc poset whose domain is a subset of ω_1 and $\dot{\mathbb{R}} \Vdash \dot{\sigma}$ is a stationary $\tau(\overrightarrow{C})$-closed subset of ω_1". Let $\dot{F}$ be a $\mathbb{P}_\alpha$-name for ω_1. Let M be a countable elementary submodel as before and set $\delta = M \cap \omega_1$. Let $q \leqslant p$ be as in the conclusion of the previous lemma. Then $q \Vdash$ "$\dot{\mathbb{R}} \Vdash \sup(\dot{C}_\delta \cap \dot{\sigma}) = \delta$". But also since q is $(M, \mathbb{P}_\alpha)$-generic, $q \Vdash \delta \in \dot{D}$. Therefore $q \Vdash$ "$\{\gamma \in \mathrm{Lim}(\omega_1)\colon \dot{\mathbb{R}} \Vdash \sup(\dot{C}_\delta \cap \dot{\sigma}) = \delta\}$ is stationary". Thus we can continue the inductive definition.

Let $\mathbb{P} = \mathbb{P}_{\omega_2}$. Let $G \subseteq \mathbb{P}$ be generic. Note that since $\mathbb{P}$ is $\aleph_2$-cc, every subset of ω_1 appears in $V[G \cap \mathbb{P}_\alpha]$ for some $\alpha < \omega_2$. First we shall show that in $V[G]$, $\vec{C}$ is a strong club guessing sequence. Let F be a club subset of ω_1 in $V[G]$. Then there exists an even $\beta < \omega_2$ such that $F = F_\beta$. Then for every $\gamma \in D_\beta$, $C_\gamma \subseteq^* F_\beta = F$. Suppose that $\mathbb{R}$ is a ccc poset whose domain is a subset of ω_1 and σ is a $\mathbb{R}$-name for a stationary $\tau(\vec{C})$-closed subset of ω_1. Then there exists an odd $\beta < \omega_2$ such that $(\mathbb{R}_\beta, \sigma_\beta) = (\mathbb{R}, \sigma)$. Then $D_\beta \subseteq \{\gamma \in \mathrm{Lim}(\omega_1)\colon\ \mathbb{R} \Vdash \sup(C_\delta \cap \sigma) = \delta\}$. But since σ is a name for a $\tau(\vec{C})$-closed set, $\mathbb{R} \Vdash \sup(C_\delta \cap \sigma) = \delta$ implies $\mathbb{R} \Vdash \delta \in \sigma$. Therefore we have $\mathbb{R} \Vdash D_\beta \subseteq \dot{\sigma}$. This concludes the proof of Proposition 8. □

Proof of Lemma 9. Let $\{\beta_n\colon n < \omega\}$ be an enumeration of all even ordinals in $M \cap \alpha$. Let $\langle \delta_n\colon n < \omega\rangle$ be an increasing cofinal sequence in δ and $\{\mathcal{D}_n\colon n \in \omega\}$ an enumeration of all open dense subsets of $\mathbb{P}_\alpha$ lying in M. Define $\dot{X}_n \in M$ as a $\mathbb{P}_\alpha$-name for $\dot{F} \cap \bigcap_{i<n}(\dot{D}_{\beta_i} \cap \dot{D}_{\beta_i+1})$. By induction, we shall construct a decreasing sequence $\langle p_n\colon n < \omega\rangle$ of elements of $\mathbb{P}_\alpha$ and a sequence $\langle \dot{E}_n\colon n < \omega\rangle$ of $\mathbb{P}_\alpha$-names such that for every $n < \omega$

(2) (a) $p_n, \dot{E}_n \in M$,
(b) $p_{n+1} \in \mathcal{D}_n$,
(c) $p_{n+1} \Vdash$ "$\dot{E}_n$ is a countable subset of $\dot{X}_n \setminus \delta_n$", and
(d) $p_{n+1} \Vdash$ "$\dot{\mathbb{R}} \Vdash \dot{\sigma} \cap \dot{E}_n \neq \emptyset$".

Let $p_0 = p$. Suppose that we have defined p_n and $\dot{E}_m$ for all $m < n$. Since p_n and $\mathcal{D}_n$ are in M, there exists a $p_{n+1} \leqslant p_n$ such that $p_{n+1} \in M \cap \mathcal{D}_n$. Since $p_{n+1}, \dot{X}_n \in M$ and $p_{n+1} \Vdash$ "$\dot{X}_n$ is club and $\dot{\mathbb{R}} \Vdash \dot{\sigma}$ is stationary", there exists a $\mathbb{P}_\alpha$-name for an $\dot{\mathbb{R}}$-name $\dot{\vartheta} \in M$ such that $p_{n+1} \Vdash$ "$\dot{\mathbb{R}} \Vdash \dot{\vartheta} \in \dot{\sigma} \cap (\dot{X}_n \setminus \delta_n)$". Since $p_{n+1} \Vdash$ "$\dot{\mathbb{R}}$ is a ccc poset", $p_{n+1} \Vdash (\exists E \in [\dot{X}_n \setminus \delta_n]^{\aleph_0})(\dot{\mathbb{R}} \Vdash \dot{\vartheta} \in E \cap \dot{\sigma})$. Let $\dot{E}_n \in M$ be a $\mathbb{P}_\alpha$-name for E. Then clearly $p_{n+1} \Vdash$ "$\dot{\mathbb{R}} \Vdash \dot{\sigma} \cap \dot{E}_n \neq \emptyset$".

Now let $c(\delta) = \{\gamma < \delta\colon\ (\exists n < \omega)(p_n \Vdash \gamma \in \bigcup_{m<\omega} \dot{E}_m)\}$ and notice that $c(\delta)$ is an unbounded subset of δ since $p_{n+1} \Vdash \delta_n \cap \dot{E}_n = \emptyset$ and

$$\mathcal{E}_n = \bigl\{q \in \mathbb{P}_\alpha\colon\ (q \perp p) \text{ or } (\exists \gamma \in \omega_1)\bigl(q \Vdash \text{ “}\dot{\mathbb{R}} \Vdash \gamma \in \dot{\sigma} \cap \dot{E}_n\text{”}\bigr)\bigr\}$$

is open dense in $\mathbb{P}_\alpha$ and $\mathcal{E}_n \in M$, so it is one of the $\mathcal{D}_m$'s we consider before and hence, for some $m \geqslant n$, p_m decides at least one element of $\dot{E}_n$ above δ_n. Define $q \in \mathbb{P}_\alpha$ by: $\mathrm{supp}(q) = M \cap \alpha$ and

$$q(0) \restriction \delta = \bigcup_{n<\omega} p_n(0),$$
$$q(0)(\delta) = \overline{c(\delta)},$$
$$q \restriction \beta \Vdash q(\beta) = \bigcup \bigl\{p_n(\beta)\colon\ n \in \omega \text{ and } \beta \in \mathrm{supp}(p_n)\bigr\} \cup \{\delta\}, \quad \text{if } \beta \in \mathrm{supp}(q).$$

Claim 10. $q \in \mathbb{P}_\alpha$.

By induction, we shall show that $q \restriction \beta \in \mathbb{P}_\beta$ for all $\beta \leqslant \alpha$. Clearly $q \restriction 1 \in \mathbb{P}_1$. Suppose that β is a limit ordinal and $q \restriction \xi \in \mathbb{P}_\xi$ for all $\xi < \beta$. Since $\mathrm{supp}(q \restriction \beta) \subseteq \mathrm{supp}(q) = M \cap \alpha$ is countable, we have $q \restriction \beta \in \mathbb{P}_\beta$. Suppose that $q \restriction \beta \in \mathbb{P}_\beta$ for $0 < \beta < \alpha$. We shall show that $q \restriction (\beta+1) \in \mathbb{P}_{\beta+1}$. It suffices to show that $q \restriction \beta \Vdash q(\beta) \in \dot{\mathbb{Q}}_\beta$. If $\beta \notin M$, then it is trivial. Hence without loss of generality, we can assume that $\beta \in M$. Then there exists an $m < \omega$ such that $\beta \in \{\beta_m, \beta_m + 1\}$. Since $p_{m+1} \Vdash (\forall n \leqslant m)(\dot{E}_n$ is countable), there exist a $\zeta < \delta$ and an $n < \omega$ such that $p_n \Vdash \bigcup_{i \leqslant m} \dot{E}_i \subseteq \zeta$. Note that $q \restriction \beta \Vdash \dot{C}_\delta = q(0)(\delta)$.

Case β is even. It suffices to show that $q \restriction \beta \Vdash \dot{C}_\delta \subseteq^* \dot{F}_\beta$. Let $\xi \in c(\delta) \setminus \zeta$. Then there exists an $n' < \omega$ such that $n \leqslant n'$ and $p_{n'} \Vdash \xi \in \dot{E}_{\tilde{n}}$ for some $\tilde{n} < \omega$. By the definition of ζ, we have $\tilde{n} > m$ and hence $p_{n+1} \Vdash \dot{X}_{\tilde{n}} \subseteq \dot{D}_\beta$. By the definition of $\dot{E}_{\tilde{n}}$, if $\max\{n', \tilde{n}\} < k < \omega$, then $p_k \Vdash \xi \in \dot{E}_{\tilde{n}} \subseteq \dot{X}_{\tilde{n}} \subseteq \dot{D}_\beta \subseteq \dot{F}_\beta$. Therefore we proved $q \restriction \beta \Vdash c(\delta) \setminus \zeta \subseteq \dot{F}_\beta$; it follows that $q \restriction \beta \Vdash \dot{C}_\delta \setminus \zeta \subseteq \dot{F}_\beta$.

Case β is odd. It suffices to show that $q \restriction \beta \Vdash$ "$\dot{\mathbb{R}}_\beta \Vdash \sup(\dot{C}_\delta \cap \dot{\sigma}_\beta) = \delta$". Let $\xi \in c(\delta) \setminus \zeta$. Then there exists an $n' < \omega$ such that $n \leqslant n'$ and $p_{n'} \Vdash \xi \in \dot{E}_{\tilde{n}}$ for some $\tilde{n}$. Then if $\max\{n', \tilde{n}\} < k < \omega$, then $p_k \Vdash \xi \in \dot{E}_{\tilde{n}} \subseteq \dot{X}_{\tilde{n}} \subseteq \dot{D}_\beta$. Since $\mathbb{P}_{\beta+1} \Vdash$ "$\dot{\mathbb{R}}_\beta \Vdash \dot{D}_\beta \subseteq \dot{\sigma}_\beta$", it follows that $p_k \Vdash$ "$\dot{\mathbb{R}}_\beta \Vdash \xi \in \dot{\sigma}_\beta$". Thus we have $q \restriction \beta \Vdash$ "$\dot{\mathbb{R}}_\beta \Vdash c(\delta) \subseteq^* \dot{\sigma}_\beta$", which clearly implies $q \restriction \beta \Vdash$ "$\dot{\mathbb{R}}_\beta \Vdash \sup(\dot{C}_\delta \cap \dot{\sigma}_\beta) = \delta$". Thus the claim is proven.

Since $q \leqslant p_n$ for all $n < \omega$, q is totally $(M, \mathbb{P}_\alpha)$-generic; actually q belongs to all open dense subsets of $\mathbb{P}_\alpha$ lying in M. By definition, we clearly have $q \Vdash \dot{C}_\delta \subseteq \dot{F}$. To show (iii), let $\zeta < \delta$ be arbitrary. Then there exists an $n < \omega$ such that $\zeta < \delta_n$. By the definition of $\dot{E}_n$, $p_{n+1} \Vdash$ "$\dot{\mathbb{R}} \Vdash \dot{\sigma} \cap \dot{E}_n \neq \emptyset$". But $q \Vdash$ "$\dot{E}_n \subseteq \dot{C}_\delta$ and $\dot{E}_n \cap \delta_n = \emptyset$". Hence we have $q \Vdash$ "$\dot{\mathbb{R}} \Vdash \sup(\dot{C}_\delta \cap \dot{\sigma}) = \delta$". Therefore q witnesses the conclusion of the lemma. □

Note that (iii) of the Lemma 9 implies that in $V^{\mathbb{P}}$, every stationary $\tau(\vec{C})$-closed subset of ω_1 contains a club subset of ω_1. By Lemma 6, $(\omega_1, \tau(\vec{C}))$ is a perfectly normal nonrealcompact space.

Now we are ready to prove our main theorem.

Theorem 11. *It is consistent with* MA + ¬CH *that there exists a perfectly normal nonrealcompact space of cardinality* $\aleph_1$.

Proof. Let $\mathbb{P}$ be the poset described in Proposition 8. Let $G \subseteq \mathbb{P}$ be generic and work in $V[G]$. Let $\vec{C}$ be defined as in Proposition 8 and let τ denote the topology $\tau(\vec{C})$. Let $\mathbb{R} = \langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\beta \colon \alpha \leqslant \omega_2;\ \beta < \omega_2 \rangle$ be a standard finite support iteration to obtain a model of MA + ¬CH, i.e., $\langle \dot{\mathbb{S}}_\alpha \colon \alpha < \omega_2 \rangle$ is an enumeration of all ccc posets of size $\leqslant \aleph_1$ which appears in some $V^{\mathbb{R}_\beta}$. It is well known that $\mathbb{R}$ is a ccc poset. Since every club subset of ω_1 added by ccc forcing contains a club subset of ω_1 lying in the ground model, $\vec{C}$ is still a strong club guessing sequence in $V[G]$. Let F be a stationary τ-closed set. Since $\mathbb{R}$ is a ccc iteration of length ω_2 with finite support, F appears in $V[G \cap \mathbb{R}_\alpha]$ for some $\alpha < \omega_2$. Thus F is a stationary τ-closed set added by $\mathbb{R}_\alpha$, which is a ccc poset of size $\aleph_1$. By the property of $\mathbb{P}$, there exists a club subset D of ω_1 such that D is contained in F. Hence every stationary τ-closed set contains a club subset of ω_1, and it follows that $\vec{C}$ is a strong τ-guessing sequence in $V^{\mathbb{P} * \mathbb{R}}$ and therefore (ω_1, τ) is perfectly normal there.

Since every τ-closed set either is non-stationary or contains a club subset, the club filter restricted to zero-sets is a non-principal countably complete z-ultrafilter. Therefore (ω_1, τ) is not realcompact. □

Hidden in the proof of Lemma 9 is a feature of the terms C_γ of the guessing sequence $\vec{C}$: they cannot be all ω-sequences. This is the main topic of the next section.

3. Destroying a strong τ-guessing sequence

The space built in the last section is a perfectly normal nonrealcompact space; this space has no τ-closed discrete stationary subspace, for if X is τ-closed and stationary, then there is a $\gamma \in X$ which is nice for X and therefore it is not an isolated point of X. The aim of this section is to show that there is a ccc poset $\mathbb{P}$ which destroys $\vec{C}$ as strong τ-guessing sequence if we assume that the order type of the sets C_γ is not γ, for a club set of $\gamma < \omega_1$.

Lemma 12. *Let $\langle E_\gamma\colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$ be a guessing sequence such that there exists an unbounded subset X of ω_1 such that $\mathrm{otp}(E_\gamma \cap \xi) < \xi$ for all $\min(E_\gamma) < \xi \leqslant \gamma \in X$. Let*

$$\mathbb{P}_0 = \{\langle p, a\rangle\colon\ p \in [\omega_1]^{<\aleph_0}\ \&\ a \in [X]^{<\aleph_0}\}$$

with the ordering $\langle p, a\rangle \leqslant \langle q, b\rangle$ if and only if $p \supseteq q$, $a \supseteq b$ and $(p \setminus q) \cap \bigcup\{E_\gamma\colon \gamma \in b\} = \emptyset$. Then $\mathbb{P}_0$ is ccc.

Proof. Let $\mathcal{A}$ be a subset of $\mathbb{P}_0$ of cardinality $\aleph_1$. We will show that $\mathcal{A}$ cannot be an antichain. Firstly, we may assume that the family of first co-ordinates of elements from $\mathcal{A}$ as well as the set of second co-ordinates of elements from $\mathcal{A}$ form Δ-systems with roots p and a, respectively. Secondly, we can also assume that for any $\alpha \in \omega_1$ we can find $\langle q, b\rangle \in \mathcal{A}$ such that $\alpha < \min((q \setminus p) \cup (b \setminus a))$. Suppose otherwise. Then either there exists a q such that $|\{b\colon \langle q, b\rangle \in \mathcal{A}\}| = \aleph_1$ or there exists a b such that $|\{q\colon \langle q, b\rangle \in \mathcal{A}\}| = \aleph_1$. In the first case, if we take two distinct b, b' so that both $\langle q, b\rangle$ and $\langle q, b'\rangle$ are in $\mathcal{A}$, then these are compatible and we are done. In the second case, it is easy to see that there exist two distinct q and q' such that both $\langle q, b\rangle$ and $\langle q', b\rangle$ are in $\mathcal{A}$ and $\max(b) < \min(q \setminus p), \min(q' \setminus p)$. Then $\langle q, b\rangle$ and $\langle q', b\rangle$ are compatible and we are done.

We shall construct inductively a sequence $\{\langle p_\alpha, a_\alpha\rangle\}_{\alpha<\omega_1}$ of conditions from $\mathcal{A}$ such that for all α, β we have

$$\max(p_\alpha \cup a_\alpha) < \min\bigl((p_\beta \setminus p) \cup (a_\beta \setminus a)\bigr), \tag{2}$$

whenever $\alpha < \beta$. By our second assumption there is no problem to choose successor terms of this sequence taking care that (2) holds. For limit β assuming $\langle p_\alpha, a_\alpha\rangle$ have been chosen for all $\alpha < \beta$, we put $\sigma(\beta) = \sup_{\alpha<\beta}(p_\alpha \cup a_\alpha)$ and choose $\langle p_\beta, a_\beta\rangle \in \mathcal{A}$ such that

$$\sigma(\beta) < \min\bigl((p_\beta \setminus p) \cup (a_\beta \setminus a)\bigr).$$

Then σ is an increasing continuous function from $\mathrm{Lim}(\omega_1)$ into ω_1. As indecomposable ordinals (i.e., those γ such that if $\xi, \zeta < \gamma$, then $\xi + \zeta < \gamma$) form a club subset of limit ordinals in ω_1, there exists an indecomposable ordinal $\beta \in \mathrm{Lim}(\omega_1)$ such that $\sigma(\beta) = \beta$. Notice that $\mathrm{otp}((\bigcup_{\gamma\in a_\beta} E_\gamma) \cap \beta) < \beta$. Since $\sigma(\beta) = \beta$, we have $\max(p_\alpha \cup a_\alpha) \leqslant \beta$ for all $\alpha < \beta$. Thus there exists an $\alpha < \beta$ such that $((p_\alpha \setminus p) \cup (a_\alpha \setminus a)) \cap \bigcup_{\gamma\in a_\beta} E_\gamma = \emptyset$. It implies that $\langle p_\alpha, a_\alpha\rangle$ and $\langle p_\beta, b_\beta\rangle$ are compatible. $\square$

Proposition 13. *Assume* MA $+ \neg$CH *and let* $\vec{E} = \langle E_\gamma\colon \gamma \in \mathrm{Lim}(\omega_1)\rangle$ *be a guessing sequence. Suppose that there exists a club subset* D *of* ω_1 *such that if* $\gamma \in D$*, then* γ *is a limit ordinal and* $\mathrm{otp}(E_\gamma) < \gamma$*. Then* $(\omega_1, \vec{E})$ *is* σ*-discrete, and hence realcompact.*

Proof. For each $\gamma \in D$, let $E'_\gamma = E_\gamma \setminus (\mathrm{otp}(E_\gamma) + 1)$. Then for every $0 < \xi \leqslant \gamma$, $\mathrm{otp}(E'_\gamma \cap \xi) < \xi$. If $\gamma \in \mathrm{Lim}(\omega_1) \setminus D$, set $E'_\gamma = E_\gamma$. Since E'_γ is an end-segment of E_γ for all $\gamma \in D$, $\tau(\langle E'_\gamma\colon \gamma \in \mathrm{Lim}(\omega_1)\rangle)$ is the same topology as $\tau(\langle E_\gamma\colon \gamma \in \mathrm{Lim}(\omega_1)\rangle)$. Thus without loss of generality, we may assume $\mathrm{otp}(E_\gamma \cap \xi) < \xi$ for every $0 < \xi \leqslant \gamma \in D$. Let $\tau = \tau(\vec{E})$.

Let $\mathbb{P}$ be a finite-support iteration of length ω of the posets defined as in Lemma 12 with $X = D$. Since $\mathbb{P}$ is a finite-support iteration of ccc posets, $\mathbb{P}$ is ccc. Let $\mathcal{D}_\xi = \{\langle p_n, a_n\rangle_{n\in\omega}\colon \xi \in p_n$ for some $n \in \omega\}$ for every $\xi < \omega_1$ and $\mathcal{E}_{m,\gamma} = \{\langle p_n, a_n\rangle_{n\in\omega}\colon \gamma \in a_m\}$ for every $m < \omega$ and $\gamma \in D$. Clearly they are dense. By applying MA, there exists a $\{\mathcal{D}_\xi\colon \xi < \omega_1\} \cup \{\mathcal{E}_{m,\gamma}\colon m < \omega$ and $\gamma \in D\}$-generic filter G in $\mathbb{P}$. Define $S_m = \bigcup\{p_m\colon \langle p_n, a_n\rangle_{n\in\omega} \in G\}$.

For every $\xi < \omega_1$, since $\mathcal{D}_\xi \cap G \neq \emptyset$, there exists an $m < \omega$ such that $\xi \in S_m$. Hence $\bigcup_{m<\omega} S_m = \omega_1$. We claim that for every $m < \omega$, any point $\gamma \in D$ is not a τ-limit point of S_m. Fix $m < \omega$ and $\gamma \in D$. By the genericity of G, there exists a $\langle p_n, a_n\rangle_{n\in\omega} \in G \cap \mathcal{E}_{m,\gamma}$. Let $\zeta = \max(p_m \cap \gamma) + 1$. Since $\gamma \in a_m$ by the definition, it is clear that for every $\langle q_n, b_n\rangle_{n\in\omega} \in G$, $(q_m \cap E_\gamma) \setminus \zeta = \emptyset$. Thus γ is not a τ-limit point of S_m.

For each $m < \omega$, let $\{T_{m,k}\colon k < \omega\}$ be a partition of S_m such that if $\gamma < \delta$ are successive points of D, then $|[\gamma, \delta) \cap T_{m,k}| \leqslant 1$ for every $k < \omega$. Then by the definition of τ, for every $m, k < \omega$, any point $\xi \in \omega_1 \setminus D$ is not a τ-limit point of $T_{m,k}$. But since we know that any point in D is not a τ-limit point of S_m, it implies that there is no τ-limit point of $T_{m,k}$, which means that $T_{m,k}$ is discrete. Therefore (ω_1, τ) is σ-discrete. □

From this proposition follows that the $\vec{C}$ obtained in Proposition 8 satisfies the condition of having its terms C_γ of order type γ, for at least a stationary set of $\gamma < \omega_1$. If $\vec{E}$ is a guessing sequence with all its terms of order type ω, our proposition is an easy consequence of a result of Devlin and Shelah [6].

4. Open questions

Part of Blair's original question remains open:

Problem 14. Is there a perfectly normal nonrealcompact space in ZFC?

We think that a stronger axiom might settle this. So we ask:

Problem 15. Does PFA imply that every perfectly normal space of cardinality less than the first measurable is realcompact? Does it do for spaces of size $\aleph_1$?

It is known that MA $+ \neg$CH implies that perfectly normal locally compact collectionwise Hausdorff spaces are paracompact, and Balogh and Junnila [3] showed that $V = L$

implies that normal locally compact spaces are collectionwise Hausdorff. If, in ZFC, X is a perfectly normal locally compact space which is not paracompact then: $\mathsf{MA} + \neg\mathsf{CH}$ implies that X is not collectionwise Hausdorff but $V = L$ implies that X is collectionwise Hausdorff. Ostaszewski's space is perfectly normal locally compact and it is not realcompact. Our space is not locally compact. We ask:

Problem 16. Is there, in ZFC, a perfectly normal locally compact space that is not realcompact? Does $\mathsf{MA} + \neg\mathsf{CH}$ imply such spaces do not exist?

The answer to the second question is affirmative if we add either locally connected or locally countable and cardinality less than $2^{\aleph_0}$ a to the required properties of the space. It is also natural to ask

Problem 17. Is there, in ZFC, a perfectly normal first countable space that is not realcompact? What can it be said under $\mathsf{MA} + \neg\mathsf{CH}$?

Another question due to Blair that we do not solve is the following. Let us note that Blair proved that the existence of a weakly perfectly normal nonrealcompact space implies the existence of a weakly perfectly normal space that is not perfectly normal [4, 4.4]. Thus under $\diamondsuit$ and consistently with $\mathsf{MA} + \neg\mathsf{CH}$ those classes of normal spaces are distinct.

Problem 18. Is there, in ZFC, a weakly perfectly normal space of cardinality less than the first measurable that is not perfectly normal?

As we noted in the introduction, under $V = L$ we do not know how to obtain guessing sequences to produce a perfectly normal topology τ on ω_1 refining the order topology and such that (ω_1, τ) is not realcompact.

Problem 19. Does $V = L$ imply there is a club guessing sequence in which the closure of each stationary set contains a club?

Finally, we would like to know:

Problem 20. Which topological properties in addition to being perfectly normal imply realcompactness?

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