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Non-trivial non weakly pseudocompact spaces

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ABSTRACT

A space Z is weakly pseudocompact if Z is G_δ-dense in at least one of its compactifications. In 1996 F.W. Eckertson [3] proposed the following problem: Find examples of Baire non Lindelöf spaces which are not weakly pseudocompact. Eckertson proposed a list of natural candidates. In this article we show that part of this list produces examples of this type by providing examples of product spaces which are Baire non-Lindelöf and not weakly pseudocompact.

1. Introduction

The notion of weak pseudocompactness, introduced in [5], is a natural generalization of a well known characterization of pseudocompactness. A space X is *weakly pseudocompact* if it is G_δ-dense in some of its compactifications.

On the one hand, every pseudocompact space is weakly pseudocompact and weak pseudocompactness is a productive property which provide us with abundant examples of weakly pseudocompact spaces. On the other hand, every weakly pseudocompact space is Baire, and weakly pseudocompact Lindelöf spaces

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are compact [5]; in other words, non Baire spaces and Lindelöf noncompact spaces are not weakly pseudocompact. Following Eckertson [3], we say that a space X is *trivially non weakly pseudocompact* if either X is Lindelöf noncompact or X is not Baire. Eckertson proved, for example, that no countable power of the Sorgenfrey line is weakly pseudocompact. But, as he pointed out, the knowledge of weakly pseudocompact spaces suffers from the scarcity of non trivial examples of non weakly pseudocompact spaces. He provided a list of spaces that "obviously" should not be weakly pseudocompact. However, as was mentioned in [10], proving weak pseudocompactness or its absence for some individual spaces turns out to be a surprisingly difficult task for some very simple spaces.

One of the spaces proposed by Eckertson is the one-point extension $D_\lambda = D \cup \{\infty\}$ of a discrete space D, where $\omega < \lambda < |D|$, topologized as follows: the points of D are isolated and the neighborhoods of ∞ are of the form $\{\infty\} \cup A$, where $A \subset D$ and $|D \setminus A| \leq \lambda$. Yet, if $\lambda^\omega = \lambda$ the authors of [10] constructed a compactification of D_λ where this space is G_δ-dense. Hence, contrary to Eckertson's expectations, this kind of spaces are in many cases weakly pseudocompact.

Another interesting kind of spaces proposed by Eckertson are the products of uncountably many Baire non-Lindelöf spaces; the cases of ω^T, $\mathbb{R}^T$ and $\mathbb{S}^T$ are particularly intriguing, where $\mathbb{S}$ is the Sorgenfrey line. We have been not able to answer the case of these particular spaces, but we clarify the situation when $X = \prod_{t\in T} X_t$ is the product of an uncountable family of Lindelöf Σ-spaces with countable π-weight and which do not admit a dense Čech-complete subspace, and when $G = \prod_{t\in T} G_t$ is the product of an uncountable family of Lindelöf Σ non Čech-complete topological groups. We show that, in both cases, these products and each one of their dense subspaces, whose projections cover all countable faces, are never weakly pseudocompact. We apply these results for some particular spaces providing examples of product spaces which are non-trivially non weakly pseudocompact spaces, as is the case of uncountable powers of Bernstein subsets of the real line. It is worth mentioning that, in the case of topological groups, some of these examples also can be obtained applying the results in [2].

After the Introduction this paper is organized as follows: Section 2 below is devoted to prove some technical results about the Stone–Čech compactification of some topological products which we use in Section 3 to prove our main results. Lastly, Section 4 provides concrete examples of non-trivial non weakly pseudocompact spaces.

As usual, the real line with the Euclidean topology is denoted by $\mathbb{R}$, and its subspace of the natural numbers will be denoted by $\mathbb{N}$. The symbol ω denotes the first infinite cardinal number. The first uncountable cardinal number is ω_1. For a space X and a subset A of X, $\mathrm{cl}_X A$ ($\mathrm{int}_X A$) will mean the closure (interior) of A in the space X. If there is no doubt as to what space X we are considering, we will simply write $\mathrm{cl}\, A$ ($\mathrm{int}\, A$) instead of $\mathrm{cl}_X A$ ($\mathrm{int}_X A$). The statement *"$X \subset Y$ is G_δ-dense in Y"* means that each nonempty G_δ-set in Y contains at least one point in X. For a set Y, $[Y]^{\leq\omega}$ and $[Y]^{<\omega}$ signify the collection of all countable subsets and of all finite subsets of Y, respectively. A *compactification* of a space X is a compact space K containing a copy of X as a dense subspace. βX denotes the Stone–Čech compactification of X. A subset N of X is *nowhere dense* in X if each nonempty open subset of X contains a nonempty open subset which misses N. A subset of X is *meager* in X if it is a countable union of nowhere dense subsets of X. A space X is *Baire* if every countable family of open dense subsets of X has a dense intersection. A collection $\mathcal{B}$ of open sets (sets with nonempty interior) in X is said to be a *π-base* (*π-pseudobase*) for X if each nonempty open subset of X contains a member of $\mathcal{B}$. A space X is *Oxtoby-complete* (*Todd-complete*) if there is a sequence $\{\mathcal{B}_n\}_{n<\omega}$ of π-bases (π-pseudobases) in X such that for any sequence $\{U_n\}_{n<\omega}$ satisfying $U_n \in \mathcal{B}_n$ and $\mathrm{cl}_X U_{n+1} \subset \mathrm{int}_X U_n$, for all $n < \omega$, we have that $\bigcap_{n<\omega} U_n \neq \emptyset$.

Throughout this article all topological spaces are considered Tychonoff.

2. The Stone–Čech compactification of an uncountable ccc product

Along this section we will consider an uncountable ccc product space $X = \prod_{t \in T} X_t$ and its Stone–Čech compactification βX.

Let us fix some notations. Assume that $A \subset T$. We set $X_A = \prod_{t \in A} X_t$. If $B \subset A$, then $p_{B,A}$ will denote the projection from X_A onto X_B, and $\pi_{B,A} : \beta X_A \to \beta X_B$ will be the continuous extension of $p_{B,A}$. Instead of $p_{A,T}$ and $\pi_{A,T}$ we will simply write p_A and π_A. Finally, consider the set $\Sigma = [T]^{\leq \omega}$.

Theorem 2.1. *If Y is a subspace of X and $p_A(Y) = X_A$ for each $A \in \Sigma$, then $\beta Y = \beta X$.*

Proof. It is sufficient to show that Y is C-embedded in X. Let $f : Y \to \mathbb{R}$ be a continuous function. Because of a Factorization Theorem by Tkachenko [1, Problem 1.7.B], there exist $A \in \Sigma$ and a continuous map $f_A : X_A \to \mathbb{R}$ such that $f = f_A \circ p_A \restriction_Y$. It happens that $f_A \circ p_A : X \to \mathbb{R}$ is the continuous extension of f. □

Lemma 2.2. *The family $\mathcal{B}_\Sigma$ consisting of all sets of the form $\pi_A^{-1}(U)$, where $A \in \Sigma$ and U is open in βX_A, is a base for βX.*

Proof. Let $C = \phi^{-1}(\mathbb{R} \setminus \{0\})$ be an arbitrary cozero set in the space βX. Another application of [1, Problem 1.7.B] shows that there exists $A \in \Sigma$ and $f_A : X_A \to \mathbb{R}$ such that $\phi \restriction_X = f_A \circ p_A$. Let $\phi_A : \beta X_A \to \mathbb{R}$ be the continuous extension of f_A. Then $\phi = \phi_A \circ \pi_A$. It follows that $C = \pi_A^{-1}(\phi_A^{-1}(\mathbb{R} \setminus \{0\})) \in \mathcal{B}_\Sigma$. Since the famiy of all cozero sets in βX is a base for βX, the family $\mathcal{B}_\Sigma$ is also a base for βX. □

The family of all nowhere dense subsets of βX will be denoted by $\mathcal{N}$. We are going to obtain some technical results concerning the family $\mathcal{N}$.

Lemma 2.3. *Let $\phi : K \to L$ be a continuous onto map and let Y be a dense subspace of K such that $\phi \restriction_Y$ is an embedding. Then the following hold:*

(1) *If $\mathcal{B}$ is a π-base for L, then $\phi^{-1}(\mathcal{B})$ is a π-base for K.*
(2) *A set $N \subset K$ is nowhere dense in K if and only if $\phi(N)$ is nowhere dense in L.*

Proof. (1) Let U be a nonempty open set in K. Select a nonempty open set O in K such that $\operatorname{cl}_K O \subset U$. Since $\phi \restriction_Y$ is an embedding, we can find an open set V in L such that $V \cap \phi(Y) = \phi(O \cap Y)$. Select an element $W \in \mathcal{B}$ such that $W \subset V$. We assert that $\phi^{-1}(W) \subset \operatorname{cl}_K O$. Indeed, otherwise we can pick $y \in Y \cap (\phi^{-1}(W) \setminus \operatorname{cl}_Y O)$ and conclude that $\phi(y) \in (V \cap \phi(Y)) \setminus \phi(O \cap Y)$, which is not possible. Therefore, we have that $\phi^{-1}(W) \subset \operatorname{cl}_K O \subset U$.

(2) Let $\mathcal{B}_L$ be a π-base for L. First, assume that N is nowhere dense in K. Because of (1), the family $\phi^{-1}(\mathcal{B}_L)$ is a π-base in K. It follows that $\mathcal{B}_K = \{U \in \phi^{-1}(\mathcal{B}_L) : U \cap N = \emptyset\}$ is a π-base in K. Now, note that $\bigcup \phi(\mathcal{B}_K)$ is a dense open subset of L and $\phi(N) \cap \bigcup \phi(\mathcal{B}_K) = \emptyset$. In this way, the set $\phi(N)$ is nowhere dense in L. Suppose now that $\phi(N)$ is nowhere dense in L. Then the family $\mathcal{B}_N = \{U \in \mathcal{B}_L : U \cap \phi(N) = \emptyset\}$ is a π-base in L. Clause (1) implies that the family $\phi^{-1}(\mathcal{B}_N)$ is a π-base in K. It follows that $\bigcup \phi^{-1}(\mathcal{B}_N)$ is open and dense in K. Moreover, note that $N \cap \bigcup \phi^{-1}(\mathcal{B}_N) = \emptyset$. Therefore, the set N is nowhere dense in K. □

Lemma 2.4. *Suppose that $\phi : K \to L$ is a continuous onto map and Y is a dense subspace of K such that $\phi \restriction_Y$ is an embedding. If C is closed in K, then there exists a nowhere dense set N in K such that $C \cup N$ is closed and ϕ-saturated in K.*

Proof. Denote by $\mathcal{N}_K$ the family of all nowhere dense sets in K. Consider the open set $U = \operatorname{int} C$ and the closed set $F := C \setminus U \in \mathcal{N}_K$. Choose a π-base $\mathcal{B}_L$ for L. By applying Lemma 2.3, we can see that the family of ϕ-saturated sets $\phi^{-1}(\mathcal{B}_L)$ is a π-base in K. Consider the family $\mathcal{B}_U = \{V \in \phi^{-1}(\mathcal{B}_L) : V \cap U = \emptyset\}$, and note that $U \cup \bigcup \mathcal{B}_U$ is open and dense in K. It follows that $N_U = K \setminus (U \cup \bigcup \mathcal{B}_U) \in \mathcal{N}_K$. The equality $U \cup N_U = K \setminus \bigcup \mathcal{B}_U$ implies that $U \cup N_U$ is closed and ϕ-saturated in K. Since $F \in \mathcal{N}_K$, from Lemma 2.3(2) we deduce that $N_F := \phi^{-1}(\phi(F)) \in \mathcal{N}_K$. As a consequence, we obtain that $N = N_U \cup N_F \in \mathcal{N}_K$ and $C \cup N$ is closed and ϕ-saturated in K. □

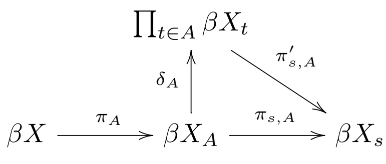
Lemma 2.5. *If $C \in \Sigma$ and $N \in \mathcal{N}$, then we can find $B \in \Sigma$ such that $C \subset B$ and $\pi_A(N)$ is nowhere dense in βX_A whenever $B \subset A \in \Sigma$.*

Proof. Choose a maximal disjoint subfamily $\mathcal{B}_N$ of $\mathcal{B}_\Sigma$ satisfying $\bigcup \mathcal{B}_N \subset \beta X \setminus N$. The maximality of $\mathcal{B}_N$, the fact that $\mathcal{B}_\Sigma$ is a base for βX, and the fact that $N \in \mathcal{N}$, imply that $\bigcup \mathcal{B}_N$ is dense in βX. Since X is dense in βX and X is ccc, the space βX also is ccc. It follows that the family $\mathcal{B}_N$ must be countable. From the definition of $\mathcal{B}_\Sigma$ we deduce that there exists $B \in \Sigma$ such that $\mathcal{B}_N$ is a family of π_B-saturated subsets of βX. We can assume that $C \subset B$. Now, choose $A \in \Sigma$ such that $B \subset A$. Observe that $\mathcal{B}_N$ is a family of π_A-saturated subsets of βX. It follows that $\bigcup \pi_A(\mathcal{B}_N)$ is an open dense subspace of βX_A which misses $\pi_A(N)$. Hence, $\pi_A(N)$ is nowhere dense in βX_A. □

From this moment until the end of this section we will assume, in addition, that $X = \prod_{t \in T} X_t$ is a product of Lindelöf Σ-spaces, and we will deal with the following objects: given $t \in T$, fix a countable family $\mathcal{C}_t$ of closed subsets of βX_t which separates points in X from points in the remainder $\beta X_t \setminus X_t$, in the sense that: for each $x \in X_t$ and $y \in \beta X_t \setminus X_t$, we can find $C \in \mathcal{C}_t$ such that $x \in C$ and $y \in \beta X_t \setminus C$. For every $A \in \Sigma$, consider the countable family $\mathcal{C}_A = \bigcup_{t \in A} \pi_t^{-1}(\mathcal{C}_t)$. Finally, select $\mathcal{C}_\Sigma = \bigcup_{A \in \Sigma} \mathcal{C}_A$.

Lemma 2.6. *If $A \in \Sigma$, $z \in \beta X$ and $\pi_A(z) \in \beta X_A \setminus X_A$, then we can find $\mathcal{C} \subset \mathcal{C}_A$ such that $X \subset \bigcup \mathcal{C}$ and $z \in \beta X \setminus \bigcup \mathcal{C}$.*

Proof. Consider the diagonal map $\delta_A = \Delta_{t \in A} \pi_{t,A} : \beta X_A \to \prod_{t \in A} \beta X_t$. Since δ_A extends the identity on X_A, we can apply Lemma 3.5.6 in [4] to see that $\delta_A(\pi_A(z)) \in \prod_{t \in A} \beta X_t \setminus X_A$. It follows that $\pi'_{s,A}(\delta_A(\pi_A(z))) \in \beta X_s \setminus X_s$ for some $s \in A$, where $\pi'_{s,A} : \prod_{t \in A} \beta X_t \to \beta X_s$ denotes the projection. Note that the following diagram is commutative:



So, $\pi_s(z) = \pi_{s,A}(\pi_A(z)) = \pi'_{s,A}(\delta_A(\pi_A(z))) \in \beta X_s \setminus X_s$. Then we can apply the fact that $\mathcal{C}_s$ separates points in X_s from points in $\beta X_s \setminus X_s$ to find a subfamily $\mathcal{C}'_s \subset \mathcal{C}_s$ such that $X_s \subset \bigcup \mathcal{C}'_s$ and $\pi_s(z) \in \beta X_s \setminus \bigcup \mathcal{C}'_s$. Setting $\mathcal{C} = \pi_s^{-1}(\mathcal{C}'_s) \subset \mathcal{C}_A$, we obtain that $X \subset \bigcup \mathcal{C}$ and $z \in \beta X \setminus \bigcup \mathcal{C}$. □

Lemma 2.7. *If we fix $N_C \in \mathcal{N}$ for each $C \in \mathcal{C}_\Sigma$, then for each $t \in T$ we can find $A \in \Sigma$ such that $t \in A$ and $\pi_A(N_C)$ is nowhere dense in βX_A for each $C \in \mathcal{C}_A$.*

Proof. Construct, inductively, an increasing sequence $\{A_n\}_{n \in \omega} \subset \Sigma$ as follows. Set $A_0 = \{t\} \in \Sigma$. Assume that A_n has been constructed. By applying Lemma 2.5, we can find $A_{n+1} \in \Sigma$ such that $A_n \subset A_{n+1}$ and $\pi_B(N_C)$ is nowhere dense in βX_B whenever $A_{n+1} \subset B \in \Sigma$ and $C \in \mathcal{C}_{A_n}$. To finish our construction, set $A = \bigcup_{n \in \omega} A_n$. We shall verify that A is as required. Indeed, given $C \in \mathcal{C}_A$, the equality $\mathcal{C}_A = \bigcup_{n \in \omega} \mathcal{C}_{A_n}$

implies that $C \in \mathcal{C}_{A_n}$ for some $n \in \omega$. Since $A_{n+1} \subset A$, our construction implies that $\pi_A(N_C)$ is nowhere dense in βX_A. □

3. Not weakly pseudocompact subspaces of some ccc products

In this section we prove the main results by providing a method to generate Baire non Lindelöf subspaces of X which are not weakly pseudocompact. Throughout this section we will assume that $X = \prod_{t \in T} X_t$ is a product of Lindelöf Σ-spaces, and Y will denote an arbitrary subspace of X satisfying $p_A(Y) = X_A$ for each $A \in \Sigma$. Moreover, we will use the constructions and notations from Section 2.

Theorem 3.1. *If X is ccc and there exists $t \in T$ such that the remainder $\beta X_A \setminus X_A$ is not meager in βX_A whenever $t \in A \in \Sigma$, then Y is not weakly pseudocompact.*

Proof. Let L be an arbitrary compactification of Y. By applying Theorem 2.1, we can find a continuous map $\phi : \beta X \to L$ such that $\phi \restriction_Y$ is an embedding. We must show that $\phi(Y)$ is not G_δ-dense in L. According to Lemma 2.4, for each $C \in \mathcal{C}_\Sigma$, we can fix a nowhere dense subset N_C of βX such that $C \cup N_C$ is closed and ϕ-saturated in βX. Because of Lemma 2.7, we can find $A \in \Sigma$ such that $t \in A$ and, if $\mathcal{N}_A = \{N_C : C \in \mathcal{C}_A\}$, then $\pi_A(\mathcal{N}_A)$ is a countable family of nowhere dense subsets of βX_A. Since $\beta X_A \setminus X_A$ is not meager in βX_A and the map π_A is onto, we can fix a point $z \in \beta X$ such that $\pi_A(z) \in \beta X_A \setminus (X_A \cup \bigcup \pi_A(\mathcal{N}_A))$. Note that $z \in \beta X \setminus \bigcup \mathcal{N}_A$. By Lemma 2.6, we can find $\mathcal{C} \subset \mathcal{C}_A$ such that $X \subset \bigcup \mathcal{C}$ and $z \in \beta X \setminus \bigcup \mathcal{C}$. We are ready to show that $\phi(Y)$ is not G_δ-dense in L. Observe that

$$\mathcal{F} = \{C \cup N_C : C \in \mathcal{C}\}$$

is a countable family of closed and ϕ-saturated subsets of K such that $Y \subset \bigcup \mathcal{F}$ and $z \in \beta X \setminus \bigcup \mathcal{F}$. As a consequence, $\phi(\mathcal{F})$ is a countable family of closed subsets of L such that $\phi(Y) \subset \bigcup \phi(\mathcal{F})$ and $\phi(z) \in L \setminus \bigcup \phi(\mathcal{F})$. Therefore, $L \setminus \bigcup \phi(\mathcal{F})$ is a nonempty G_δ-subset of L which misses $\phi(Y)$. □

Theorem 3.2. *Suppose that X_t has countable π-weight for each $t \in T$. If Y is weakly pseudocompact, then each X_t contains a dense Čech-complete subspace.*

Proof. Assume that there exists $t \in T$ such that X_t does not contain a dense Čech-complete subspace. We will verify that X satisfies the conditions in Theorem 3.1, contradicting the fact that Y is weakly pseudocompact. Observe that a topological space has a meager remainder if and only if it contains a dense Čech-complete subspace. Thus, $\beta X_t \setminus X_t$ is not meager in βX_t. Assume that $t \in A \in \Sigma$. Set $B = A \setminus \{t\}$ and note that X_B has a countable π-base. Then we can use Theorem 3.9(c) in [6] to see that βX_B has a countable π-base. Now, we can apply Theorem 1 in [11] to see that $(\beta X_t \setminus X_t) \times \beta X_B$ is not meager in $\beta X_t \times \beta X_B$. Since $(\beta X_t \setminus X_t) \times \beta X_B \subset (\beta X_t \times \beta X_B) \setminus X_A$, the remainder $(\beta X_t \times \beta X_B) \setminus X^A$ is not meager in $\beta X_t \times \beta X_B$. From Lemma 2.3(2) and Lemma 3.5.6 in [4], we can deduce that $\beta X_A \setminus X_A$ is not meager in βX_A. Finally, the space X, being a product of spaces with a countable π-base, has the ccc property (see (2.7) in [11]). Theorem 3.1 implies that Y is not weakly pseudocompact, contradicting our initial assumption. Therefore, each X_t contains a dense Čech-complete subspace. □

Theorem 3.3. *Suppose that X_t is Baire, non-compact, and has countable π-weight for each $t \in T$. Then Y is a Baire space which is not Lindelöf.*

Proof. First we are going to verify that Y is a Baire space. We know that each X_t is a Baire space with a countable π-base, so we can apply Theorem 3 from [11] to see that X has the Baire property. Observe that

Y is G_δ-dense in X. Since the Baire property is inherited by G_δ-dense subspaces, we conclude that Y has the Baire property.

Recall that T is uncountable. Fix an uncountable collection $A = \{t_\alpha : \alpha < \omega_1\} \subset T$ and set $A_\alpha = \{t_\beta : \beta < \alpha\}$ for each $\alpha \leq \omega_1$. To see that Y is not Lindelöf, it is enough to show that $p_A(Y) \subset X_A$ is not Lindelöf. Given $t \in T$, since X_t is Lindelöf and not compact, it is not pseudocompact. So, if $p_A(Y) = X_A$, then $p_A(Y)$ contains a closed copy of $\mathbb{N}^{\omega_1}$ and hence it is not Lindelöf. Assume now, on the other hand, that there exists a point $x \in X_A \setminus p_A(Y)$. For each $\alpha < \omega_1$, since $p_{A_\alpha}(Y) = X_{A_\alpha}$, we can select $y_\alpha \in p_A(Y)$ such that $p_{A_\alpha,A}(y_\alpha) = p_{A_\alpha,A}(x)$. Let $D = \{y_\alpha\}_{\alpha<\omega_1}$. Choose $y \in p_A(Y)$ arbitrarily. Then $y \neq x$ and, because of this, we can find $t \in A$ such that $y(t) \neq x(t)$. Note that the neighborhood $U_y = \{z \in X_A : z(t) \neq x(t)\}$ of y contains at most countably many elements of D. Therefore, D is an uncountable subset of $p_A(Y)$ without complete accumulation points in $p_A(Y)$. Hence, $p_A(Y)$ is not a Lindelöf space. □

As we mentioned before, given a space Z, the space Z does not contain a dense Čech-complete subspace if and only if it has a non meager remainder in some compactification. Besides, if Z has countable pseudocharacter, then the following conditions on a subset Y of Z^T are equivalent:

(i) Y is G_δ-dense in Z^T.
(ii) $p_A(Y) = Z^A$ for each $A \in \Sigma$.

So, we arrive to the following corollary of the previous theorems:

Corollary 3.4. *Let Z be a separable metrizable space with a non meager remainder in some compactification. If T is uncountable and Y is a G_δ-dense subspace of Z^T, then Y is a Baire space which is neither Lindelöf nor weakly pseudocompact.*

Every Oxtoby-complete space is Todd-complete, and every weakly pseudocompact space is Todd-complete [12]. It is still unknown whether every Todd-complete space is Oxtoby-complete, and whether every weakly pseudocompact space is Oxtoby-complete. It is worth mentioning, with respect to Theorem 3.2 above, that if G_t is a topological group with countable base for each $t \in T$, and H is an Oxtoby-complete G_δ-dense subgroup of $\prod_{t\in T} G_t$, then each G_t has to be Čech-complete (see Theorem 10.1 in [2]).

Theorem 3.5. *Let $G = \prod_{t\in T} G_t$ be a product of Lindelöf Σ topological groups. If Y is a weakly pseudocompact subspace of G such that $p_A(Y) = \prod_{t\in A} G_t$ for each $A \in \Sigma$, then G_t is Čech-complete for all $t \in T$.*

Proof. Let $G_A = \prod_{t\in A} G_t$ for each $A \in \Sigma$. Assume that G_t is not Čech-complete for some $t \in T$. Proceeding as in the proof of Theorem 3.2, we will verify that X satisfies the conditions in Theorem 3.1, contradicting the fact that Y is weakly pseudocompact. Assume that $t \in A \in \Sigma$. Since Čech-completeness is invariant under continuous open images in the class of topological groups [8, Lemma 4.5], we deduce that G_A is not Čech-complete. It follows from [8, Corollary 4.4] that G_A does not contain any Čech-complete subspace. In an equivalent way, the remainder $\beta G_A \setminus G_A$ is not meager in βG_A. Finally, given a set $F \in [T]^{<\omega}$, the topological group G_F is Lindelöf Σ. So, we can apply a Theorem by Uspenskii in [13], to see that G_F has the ccc property. It follows from a well known result that the product G also has the ccc property. Because of Theorem 3.1, the space Y is not weakly pseudocompact, contradicting our initial assumption. Therefore, the topological group G_t is Čech-complete for all $t \in T$. □

4. Examples

We conclude this work by presenting some examples.

Example 4.1. Let X be a Bernstein subset of the real line $\mathbb{R}$; that is, a subset of the real line which intersects P and $\mathbb{R} \setminus P$ for every perfect subset P of $\mathbb{R}$. Bernstein subsets of the real line can be constructed in a standard way using the Axiom of Choice. From the definition it is clear that $\mathbb{R} \setminus X$ also is a Bernstein set. Since every countable intersection of open dense subspaces of $\mathbb{R}$ contains a perfect set, both X and $\mathbb{R} \setminus X$ have the Baire property. Consider the one point compactification S of $\mathbb{R}$. Note that the compactification S witnesses that X satisfies the hypotheses from Corollary 3.4. So, we conclude that, if Y is a G_δ-dense subspace of X^T, then Y is a Baire space which is neither Lindelöf nor weakly pseudocompact.

Example 4.2. Let ϕ be an ultrafilter in $\beta\mathbb{N} \setminus \mathbb{N}$. Let us denote by $\mathbb{N}_\phi$ the subspace $\mathbb{N} \cup \{\phi\}$ of $\beta\mathbb{N}$. It was proved in [9] that the space of all continuous real valued functions $X = C_p(\mathbb{N}_\phi)$ on $\mathbb{N}_\phi$ is a Baire space. Since X is contained in $\mathbb{R}^{\mathbb{N}_\phi}$ and $\mathbb{N}_\phi$ is not discrete, we can fix a map $f \in \mathbb{R}^{\mathbb{N}_\phi} \setminus X$. Let us observe that $f + X$ is a dense subset of $\mathbb{R}^{\mathbb{N}_\phi} \setminus X$ with the Baire property. Besides $(f + X) \cap X = \emptyset$. Hence, both X and $\mathbb{R}^{\mathbb{N}_\phi} \setminus X$ have the Baire property. Consider the one point compactification S of $\mathbb{R}$. Note that the compactification $S^{\mathbb{N}_\phi}$ of X witnesses that X satisfies the hypotheses from Corollary 3.4. As before, we conclude that: if Y is a G_δ-dense subspace of X^T, then Y is a Baire space which is neither Lindelöf nor weakly pseudocompact. Note that in many cases Y is a topological group.

Example 4.3. Let E be an infinite dimensional second countable Baire topological vector space. Choose a Hamel basis B for E and let $\{B_n\}_{n\in\omega}$ be a partition of B in infinite sets. Denote by X_n the linear subspace of E generated by $\bigcup_{m\leq n} B_m$. We know that $E = \bigcup_{n\in\omega} X_n$. Since E has the Baire property, there exists $n \in \omega$ such that $X = X_n$ is not meager. By applying Proposition 2.1 in [7], we can see that X actually has the Baire property. Besides, it follows from Proposition 2.3 in [7] that X is a dense subspace of E. On the other hand, by taking a point $e \in X_{n+1} \setminus X$, we can see that $e + X$ is a Baire dense subspace of E contained in $E \setminus X$. Hence $E \setminus X$ is also a Baire space. Therefore every second countable compactification M of E witnesses that X satisfies the hypotheses from Corollary 3.4. Again, we conclude that: if Y is a G_δ-dense subspace of X^T, then Y is a Baire space which is neither Lindelöf nor weakly pseudocompact. As before, in many cases Y is a topological group.

Remark 4.4. Example 4.2 produces a C_p-space which is a non-trivial non weakly pseudocompact space. Indeed, $C_p(\mathbb{N}_\phi)^{\omega_1} \cong C_p(\oplus_{\xi<\omega_1}\mathbb{N}_\phi)$ is such an example.

The following question arises naturally.

Question 4.5. Let X be a Lindelöf Σ-space with countable π-weight and containing a dense Čech-complete subspace. Can an uncountable power of X be weakly pseudocompact?

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