\section{Notations, basic definitions and introduction}

Throughout this article all topological spaces are considered Tychonoff and with more than one point if the contrary is not specified. As usual, the real line with the Euclidean topology is denoted by $\mathbb{R}$, and its subspace of integers by $\mathbb{Z}$. The symbol $\omega$ denotes the first infinite cardinal number and $\mathbb{N}$ the natural numbers; that is, $\mathbb{N} = \omega \setminus \{0\}$. The first uncountable cardinal number is $\omega_1$. For a space $X$ and a subset $H$ of $X$, $\text{cl}_X H$ and $\text{int}_X H$ will mean the closure and the interior of $H$ in the space $X$. If there is no doubt as to what space $X$ we are considering, we will simply write $\text{cl} H$ and $\text{int} H$ instead of $\text{cl}_X H$ and $\text{int}_X H$. For a set $Y$, $[Y]<\omega$, $[Y]\leq\omega$ and $[Y]\omega$ signify the collection of finite subsets, countable subsets and infinite countable subsets of $Y$, respectively.

We are going to denote by $C_p(X,Y)$ the space of continuous functions from $X$ to $Y$ with the topology of the pointwise convergence, which is that inherited by the product topology in $Y^X$; that is, the topology in $C_p(X,Y)$ is the topology generated by all the sets of the form:

$$[x_1, \ldots, x_n; B_1, \ldots, B_n] = \{f \in C_p(X,Y) : f(x_i) \in B_i, i = 1, \ldots, n\}$$

where $n \in \omega$, $\{x_1, \ldots, x_n\} \subseteq X$, and $B_1, \ldots, B_n$ are open subsets of $Y$. When $Y$ is the real line with its usual topology, we write $C_p(X)$ instead of $C_p(X,\mathbb{R})$. For every space of the form $C_p(X,Y)$ considered in this article, the spaces $X$ and $Y$ are such that $C_p(X,Y)$ is dense in $Y^X$.

A compactification of a space $X$ is a compact space $K$ containing $X$ as a dense subspace. The statement “$X \subseteq Y$ is $G_\delta$-dense in $Y$” means that each nonempty $G_\delta$-set in $Y$ contains at least one point in $X$.

A space $X$ is weakly pseudocompact if it is $G_\delta$-dense in some of its compactifications. Every pseudocompact space is weakly pseudocompact; weak pseudocompactness is a productive property; and every weakly pseudocompact space is a Baire space (see [8]). F.W. Eckertson posed back in 1996 in [6] the following problems:

\textbf{Problem 1.1.} \hspace{5mm}
(1) Is $\mathbb{R}^\kappa$, $\mathbb{Z}^\kappa$ or $\mathbb{S}^\kappa$ weakly pseudocompact when $S$ is the Sorgenfrey line and $\kappa > \omega$?

(2) Is a $\Sigma$-product of $\mathbb{R}^\kappa$, $\mathbb{Z}^\kappa$ or $\mathbb{S}^\kappa$ weakly pseudocompact?

A more general problem is:

\textbf{Problem 1.2.} Characterize spaces $X$ and $Y$ for which the space of continuous functions from $X$ to $Y$ with the pointwise convergence topology, $C_p(X,Y)$, is weakly pseudocompact.

Some contributions in the direction of solving this problem are made in [2], [3], [4], [5] and [9]. In this article, we continue these efforts and prove the following:

\textbf{Abstract.} A space $Z$ is weakly pseudocompact if $Z$ is $G_\delta$-dense in at least one of its compactifications. F.W. Eckertson asked in 1996 [6] if the spaces $\mathbb{R}^{\omega_1}$, $\mathbb{N}^{\omega_1}$ or $\mathbb{S}^{\omega_1}$ are weakly pseudocompact where $S$ is the Sorgenfrey line. In [5] a more general question was posed: What conditions must spaces $X$ and $G$ satisfy so that the space $C_p(X,G)$ be weakly pseudocompact?

In this paper we answer Eckertson’s question in the negative and we prove that $C_p(X,G)$ is never weakly pseudocompact if $G$ is a metrizable separable locally compact non compact topological group, assuming that $C_p(X,G)$ is dense in $G^X$. In particular, $C_p(X)$ is not weakly pseudocompact for any Tychonoff space $X$, and $C_p(X,Z)$ is not weakly pseudocompact if $X$ is zero-dimensional $T_2$. 

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Theorem 1.3. Let $G$ be a metrizable separable locally compact and non compact topological group. Then, the space $C_p(X, G)$ is not weakly pseudocompact.

Numerous outstanding consequences arise from this result. In particular we obtain a negative answer to items (1) and (2) in Problem 1.1, and determine a wide range class of topological spaces $X$ and $G$ for which Problem 1.2 has a negative answer. Furthermore, we show that for every cardinal number $\kappa$, there is an $\mathbb{R}$-factorizable topological group with weight $\kappa \cdot \omega$ which is Sánchez-Okunev countably compact, Oxtoby countably compact and non weakly pseudocompact, answering a part of a question posed in [4].

Section 2 below is devoted to presenting the concepts and first lemmas that we are going to use in order to prove Theorem 1.3. In Section 3, we give some properties about nowhere dense subsets in the limit of an inverse system of compact spaces. In Section 4 we give some base assignments for the spaces introduced in Section 2. In Section 5 we discuss the relations given by a condensation $c : S \to X$ between the inverse systems defined by $S$ and $X$ as constructed in Section 2. Section 6 is devoted to proving our main result, and some of its consequences are listed in Section 7. Finally, in Section 8, we present some open problems.

2. FIRST NOTATIONS, DEFINITIONS AND LEMMAS

Let $X$ be a Tychonoff space and let $Y$ be an infinite set. Assume that $D \subseteq A \subseteq Y$. We will denote by $p_{D,A}$ the projection from $X^A$ onto $X^D$. Let $\pi_{D,A} : \beta(X^A) \to \beta(X^D)$ be the continuous extension of $p_{D,A}$. We set $P = \prod_{G \in [Y]^{\leq \omega}} \beta(X^G)$. For each $A \in [Y]^{\leq \omega}$, $p_A$ is the projection from $P$ onto $\beta(X^A)$. Consider the diagonal map $d = \Delta_{G \in [Y]^{\leq \omega}} p_{G,Y} : X^Y \to P$ and let $W = d(X^Y)$. Moreover, set $K := \{ f \in P : \pi_{D,A}(p_A(f)) = p_D(f) \text{ whenever } D \subseteq A \in [Y]^{\leq \omega} \}$. Finally put $q_A = p_A \upharpoonright K$ for all $A \in [Y]^{\leq \omega}$.

Remark 2.1. Assume that $C \subseteq D \subseteq A \subseteq [Y]^{\leq \omega}$. Then we have the following equalities:

(a) $\pi_{D,A} \circ q_A = q_D$;

(b) $\pi_{C,A} = \pi_{C,D} \circ \pi_{D,A}$; and

(c) $p_{A,Y} = q_A \circ d$.

Proof. (a) If $f \in K$, then $\pi_{D,A}(q_A(f)) = \pi_{D,A}(p_A(f)) = p_D(f) = q_D(f)$. Therefore $\pi_{D,A} \circ q_A = q_D$.

(b) Since the functions $p_{C,A}$, $p_{C,D}$ and $p_{D,A}$ are projections, we have that $p_{C,A} = p_{C,D} \circ p_{D,A}$. Moreover, $\pi_{C,A}$ and $\pi_{C,D} \circ \pi_{D,A}$ are extensions of $p_{C,A}$ and $p_{C,D} \circ p_{D,A}$ from $X^A$ on $\beta(X^A)$, respectively. Therefore, $\pi_{C,A} = \pi_{C,D} \circ \pi_{D,A}$.

(c) For each $f \in X^Y$ observe that $q_A \circ d(f) = q_A(d(f)) = p_A(d(f)) = p_{A,Y}(f)$. So, $p_{A,Y} = q_A \circ d$. □

Diagram 1. All the circles are commutative.

Remark 2.2. Because of the above mentioned definitions and Remark 2.1(b), the triplet $\mathcal{M} = \{ \beta(X^A), \pi_{D,A}, [Y]^{\leq \omega} \}$ is an inverse system, where $[Y]^{\leq \omega}$ is directed by the relation $\subseteq$. Besides, $K$ is the limit of the inverse system $\mathcal{M}$. (See [7], page 98.)
Remark 2.3. The space $K$ is compact.

Proof. Since $K$ is the limit of the inverse system $\mathfrak{M}$ and $\beta(X^A)$ is Hausdorff for each $A \in [Y]^{\leq \omega}$, the space $K$ is closed in $P$ (see [7, Proposition 2.5.1]). Thus, $K$ is compact. □

Lemma 2.4. The family of all sets of the form $q_A^{-1}(U)$ where $A \in [Y]^{\leq \omega}$ and $U$ is open in $\beta(X^A)$ is a base for the topology on $K$.

Proof. Let $f \in K$, and let $O$ be a canonical open set in $P$ containing $f$. Then, there are $n < \omega$, $G_0, \ldots, G_n \in [Y]^{\leq \omega}$ and open subsets $B_0, \ldots, B_n$ of $\beta(X^{G_0}), \ldots, \beta(X^{G_n})$, respectively, such that $O = \bigcap_{i \leq n} p_{G_i}^{-1}(B_i)$. Take $A = \bigcup_{i \leq n} G_i$ and consider the open subset $U := \bigcap_{i \leq n} p_{G_i}^{-1}(B_i)$ of $\beta(X^A)$.

We claim that $f \in q_A^{-1}(U) \subseteq O \cap K$. Indeed, $\pi_{G_i,A}(q_A(f)) = q_{G_i}(f) = p_{G_i}(f) = B_i$ for each $i \leq n$ and then $f \in q_A^{-1}(U)$. Now, by using Remark 2.1 we obtain that $q_A^{-1}(U) = q_A^{-1}(\bigcap_{i \leq n} \pi_{G_i,A}^{-1}(B_i)) = \bigcap_{i \leq n} q_A^{-1}(\pi_{G_i,A}^{-1}(B_i)) = \bigcap_{i \leq n} q_A^{-1}(B_i) = \bigcap_{i \leq n} p_{G_i}^{-1}(B_i) \cap K = O \cap K$. □

Lemma 2.5. The family $\{p_{G,Y} : G \in [Y]^{\leq \omega}\}$ separates points as well as points from closed sets in $XY$.

Proof. Since $XY$ is $T_0$, it is enough to prove that $\{p_{G,Y} : G \in [Y]^{\leq \omega}\}$ separates points from closed sets. Pick a point $f$ and a closed set $F$ such that $f \notin F$. Choose a canonical open set $O := \bigcap_{i \leq n} p_{G_i,Y}^{-1}(B_i)$ in $XY$ such that $f \in O$ and $O \cap F = \emptyset$. Take $G = \{y_0, \ldots, y_n\}$. Note that $p_{G,Y}(f) \in p_{G,Y}(O)$. The fact that $p_{G,Y}(p_{G,Y}(O)) = O$ implies that $p_{G,Y}(p_{G,Y}(O)) \cap F = \emptyset$ and hence $p_{G,Y}(O) \cap p_{G,Y}(F) = \emptyset$. Therefore $p_{G,Y}(f) \notin \text{cl}(p_{G,Y}(F))$. □

Because of Theorem 2.3.20 in [7] and Lemma 2.5, we obtain:

Corollary 2.6. The map $d : XY \to W$ is an embedding.

Remark 2.7. The space $K$ contains $W$ as a dense subspace.

Proof. First of all we have to prove that $W \subseteq K$. Pick $g \in W$. Choose $f \in XY$ such that $g = d(f)$. Since $\pi_{D,A}(p_A(g)) = \pi_{D,A}(p_A(d(f))) = \pi_{D,A}(p_A(f)) = p_D,A(p_A(f)) = p_D,Y(f) = p_D(d(f)) = p_D(g)$ whenever $D \subseteq A \in [Y]^{\leq \omega}$, we obtain that $g \in K$.

Because of Lemma 2.4, to show that $W$ is dense in $K$ it is enough to verify that $W \cap q_A^{-1}(U) \neq \emptyset$ for each $A \in [Y]^{\leq \omega}$ and every open subset $U$ of $\beta(X^A)$. Pick $A \in [Y]^{\leq \omega}$ and an open subset $U$ of $\beta(X^A)$. Select $f \in p_{A,Y}^{-1}(U \cap X^A) \subseteq XY$. Then $q_A(d(f)) = p_{A,Y}(f) \in U \cap X^A \subseteq U$ and therefore $d(f) \in W \cap q_A^{-1}(U)$. □

Remark 2.8. For each $A \in [Y]^{\leq \omega}$ the function $q_A : K \to \beta(X^A)$ is onto.

Proof. Since $q_A(W) = q_A(d(X^Y)) = p_{A,Y}(X^Y) = X^A$, the continuity of $q_A$ and the compactness of $K$ imply that the function $q_A$ maps $K$ onto $\beta(X^A)$. □

Lemma 2.9. If $XY$ has countable cellularity and $C \subseteq XY$ is $G_\delta$-dense, then $K$ is the Stone-Čech compactification of $d(C)$.

Proof. We will prove that $K$ satisfies the extension property which characterizes $\beta(d(C))$. Let $\phi : d(C) \to I$ be a continuous map, where $I$ is the unit interval. By the Tkachenko Factorization Theorem [1, Problem 1.7.B], there exists $A \in [Y]^{\leq \omega}$ and a map $\phi_A : p_{A,Y}(C) \to I$ such that $\phi \circ d|_C = \phi_A \circ p_{A,Y}|_C$. Let $\psi_A : \beta(p_{A,Y}(C)) \to I$ be the continuous extension of $\phi_A$. By hypothesis $C$ is $G_\delta$-dense in $XY$, so $\beta(p_{A,Y}(C)) = \beta(X^A) = q_A(K)$. Thus the composition $\psi_A \circ q_A : K \to I$ makes sense. Let us verify that the map $\psi_A \circ q_A$ is the desired continuous extension of $\phi$. Indeed, given $f \in d(C)$ select $g \in C$ such that $f = d(g)$ and observe that $\psi_A \circ q_A(f) = \psi_A \circ q_A \circ d(g) = \psi_A \circ p_{A,Y}(g) = \phi_A \circ p_{A,Y}(g) = \phi \circ d(g) = \phi(f)$. Therefore, $K$ is the Stone-Čech compactification of $d(C)$. □

3. Nowhere dense subsets of $K$

Denote by $\mathcal{N}$ the family of all nowhere dense subsets of the inverse limit $K$. We are going to obtain some lemmas which will play a crucial role when we prove the main theorem of this article (Theorem 6.1, below). These lemmas have to do with some relations concerning to the family $\mathcal{N}$.

Remark 3.1. Let $\phi : K \to L$ be a continuous map and let $Z$ be a dense subspace of $K$ such that $\phi|_Z$ is an embedding. Then $\phi^{-1}(\phi(\mathcal{N})) \subseteq \mathcal{N}$. □
Proof. Fix $N \in \mathcal{N}$. We can suppose that $N$ is closed in $K$. Let $U$ be the interior of the closed set $\phi^{-1}(\phi(N))$ in $K$. Assume that $z \in Z \cap U$. Since $\phi(z) \in \phi(N)$, there is $x \in N$ such that $\phi(z) = \phi(x)$, and hence $z = x \in N$ (see p. 168 in [7, Lemma 3.5]). As $z$ was arbitrary, it must happen that $U \subseteq \text{cl}(Z \cap U) \subseteq N$. It follows that $U = \emptyset$. Therefore $\phi^{-1}(\phi(N)) \subseteq N$.

\begin{lemma}
Suppose that $\phi : K \rightarrow L$ is a quotient map and $Z$ is a dense subspace of $K$ such that $\phi \upharpoonright Z$ is an embedding. If $U$ is open in $K$, then there exists $N \in \mathcal{N}$ such that $U \cap N$ is closed and $\phi$-saturated in $K$.
\end{lemma}

Proof. Note that since $\phi \upharpoonright Z$ is an embedding, we must have $\phi^{-1}(\phi(x)) = \{x\}$ for each $x \in Z$.

Since $\phi(U \cap Z)$ is open in $\phi(Z)$, we can find an open set $O$ in $L$ such that $O \cap \phi(Z) = \phi(U \cap Z)$. Consider the $\phi$-saturated and open set $V = \phi^{-1}(O)$. Note that $V \cap Z = U \cap Z$. Indeed, the relation $U \cap Z \subseteq V \cap Z$ is obvious. Now if $f \in V \cap Z$ then $\phi(f) \in O \cap \phi(Z) = \phi(U \cap Z)$. This means that there is $g \in U \cap Z$ such that $\phi(g) = \phi(f)$. The fact that $g \in Z$ implies that $g = f$; so, $f \in U \cap Z$. Now, note that the equality $V \cap Z = U \cap Z$ implies that $\text{cl}(U) = \text{cl}(V) = \text{cl}(U \cap V)$.

We claim that the set $N = \phi^{-1}(\phi(\text{cl}(U) \setminus (U \cap V)))$ is as we need. On the one hand, note that $\text{cl}(U \cap V) = \text{cl}(U \cup V) \setminus (U \cap V) \subseteq N$. It follows from Remark 3.1 that $N \in \mathcal{N}$. On the other hand, note that $U \cap N \subseteq \phi^{-1}(\phi(\text{cl}(U))) = \phi^{-1}(\phi(\text{cl}(V))) = \phi^{-1}(\phi(\text{cl}(V) \setminus V)) = \phi^{-1}(\phi(V)) \cup \phi^{-1}(\phi(\text{cl}(V) \setminus V)) = V \cup \phi^{-1}(\phi(\text{cl}(U) \setminus V)) \subseteq V \cup N \subseteq \text{cl}(U) \cup N \subseteq U \cap N$.

Therefore $U \cap N = \phi^{-1}(\phi(\text{cl}(U)))$ is closed and $\phi$-saturated in $K$.

\begin{lemma}
Assume that for each $A \in [Y]^{<\omega}$ there exist $N_A \in [N]^{<\omega}$ and $B_A \in [\tau(\beta(X^A))]^{<\omega}$ satisfying the following conditions:

(a) $N_A = \bigcup_{F \in [A]^{<\omega}} N_F$;

(b) $B_A = \bigcup_{F \in [A]^{<\omega}} \pi_{A,F}^{-1}(B_F)$ and $B_A$ is a $\pi$-base for $\beta(X^A)$; and

(c) $B_K = \bigcup_{F \in [\mathcal{L}]^{<\omega}} \pi_{F}^{-1}(B_F)$ is a $\pi$-base for $K$.

Then there exists $A \in [Y]^{<\omega}$ such that $q_A(N)$ is nowhere dense in $\beta(X^A)$ for each $N \in N_A$.
\end{lemma}

Proof. For each $B \in B_K$ fix $F_B \in [Y]^{<\omega}$ such that $B \subseteq q^{-1}_F(B_F)$. Let $\mathcal{M}$ be a countable elementary submodel of $H(\theta)$ for some large enough cardinal $\theta$ which contains everything relevant for this proof, in particular the assignments from the hypotheses. Consider the set $A = \bigcup (\mathcal{M} \cap [Y]^{<\omega}) \subseteq [Y]^{<\omega}$. Note that $q^{-1}_A(B_A) = \bigcup_{F \in [A]^{<\omega}} q^{-1}_A(\pi_{A,F}^{-1}(B_F)) = \bigcup_{F \in [A]^{<\omega}} q^{-1}_F(B_F) = B_K \cap \mathcal{M}$. Choose $N \in N_A$. The clause (a) implies that $N \subseteq N_F$ for some $F \in [A]^{<\omega}$. It follows that $N \subseteq \mathcal{M} \cap N$.

By using the conditions (b) and (c) we obtain that

$$\forall U \subseteq B_K \exists V \in B_K (V \subseteq U \cup V \cap N = \emptyset) \Leftrightarrow \forall U \subseteq (B_K \cap \mathcal{M}) \exists V \in (B_K \cap \mathcal{M}) (V \subseteq U \cup V \cap N = \emptyset) \Leftrightarrow \forall U \subseteq q^{-1}_A(B_A) \exists V \in q^{-1}_A(B_A) (V \subseteq U \cup V \cap N = \emptyset) \Leftrightarrow \forall U \subseteq B_A \exists V \in B_A (V \subseteq U \cup V \cap N = \emptyset).$$

From these equivalences we conclude that $q_A(N)$ is nowhere dense in $\beta(X^A)$.

\begin{remark}
Let $e : S \rightarrow X$ be a continuous onto map and let $B$ be a family of open subsets of $X$ such that $e^{-1}(B)$ is a $\pi$-base for $S$. If $N$ is nowhere dense in $S$, then $e(N)$ is nowhere dense in $X$.
\end{remark}

Proof. Let $N$ be a nowhere dense set in $S$. Choose an arbitrary nonempty open subset $U$ of $X$. By our hypothesis, the open set $e^{-1}(U)$ contains a member $e^{-1}(B)$ of $e^{-1}(B)$ such that $e^{-1}(B) \cap N = \emptyset$. Since the map $e$ is onto, we must have that $B \subseteq U$. Besides, it is clear that $B \cap e(N) = \emptyset$. In this way, $e(N)$ is nowhere dense in $X$.

4. The spaces $W$ and $K$ when $X$ is locally compact and non compact

In this section we determine external bases for $W$ in $K$ and $X^A$ in $\beta X^A$, obtained from a given base of $X$, for the case that $X$ is locally compact and non compact. First we recall some basic facts.

\begin{remark}
Let $D$ be a dense subspace of $X$ and let $B$ be a collection of open subsets of $X$. If $B \cap D$ is a base (resp., a $\pi$-base) for $D$, then $B$ contains a local base for each point of $D$ in $X$ (resp., is a $\pi$-base for $X$).
\end{remark}

Proof. Select $d \in D$. Let $U$ be an open set in $X$ containing $d$. Choose an open set $O$ in $X$ such that $d \in O \subseteq \text{cl}_X O \subseteq U$. It happens that there is an element $B$ in $B$ such that $d \in B \cap D \subseteq O$. Then, $d \in B \subseteq \text{cl}_X B = \text{cl}_X (B \cap D) \subseteq \text{cl}_X O \subseteq U$. In the same way we can prove the case of $\pi$-bases.
Remark 4.2. Let \( c : S \to X \) be a condensation and let \( e : \beta S \to \beta X \) be the continuous extension of \( c \). If \( B \) is a family of open subsets of \( \beta X \) and \( e^{-1}(B \cap X) \) is a \( \pi \)-base for \( S \), then \( e^{-1}(B) \) is a \( \pi \)-base for \( \beta S \).

**Proof.** Given an open set \( B \) in \( \beta X \) note that \( e^{-1}(B) \cap \beta S = e^{-1}(B \cap X) \). Hence \( e^{-1}(B) \cap \beta S = e^{-1}(B \cap X) \). It follows that \( e^{-1}(B) \cap S = e^{-1}(B \cap X) \) is a \( \pi \)-base for \( S \). Therefore, we can apply Remark 4.1 to see that \( e^{-1}(B) \) is a \( \pi \)-base for \( \beta S \). \( \square \)

Assume that the space \( X \) is locally compact and non compact. Fix a base \( B \) for the space \( X \) constituted by open subsets with compact closure. For each \( F \in [Y]^{<\omega} \) and \( s \in B^F \) we set \( \bar{s} = \bigcap_{y \in F} p_y^{-1}(s(y)) \), where \( p_y : X^F \to X \) is the \( y \)-projection. Given \( F \in [Y]^{<\omega} \) and \( s \in B^F \) note that since the closure of each element of \( B \) is compact, the set \( \bar{s} \) has a compact closure in \( X^F \) and is open in \( \beta(X^F) \); and as a consequence \( \pi_{F,A}(\bar{s}) \) and \( q_F^{-1}(\bar{s}) \) are open in \( \beta(X^A) \) and \( K \), respectively, whenever \( F \subseteq A \subseteq Y \). Consider the next families of open sets:

\[
B_K = \{ q_F^{-1}(\bar{s}) : F \in [Y]^{<\omega} \text{ and } s \in B^F \}
\]
and

\[
B_A = \{ \pi_{F,A}^{-1}(\bar{s}) : F \in [A]^{<\omega} \text{ and } s \in B^F \},
\]
for each \( A \in [Y]^{<\omega} \).

**Remark 4.3.** Suppose that \( A \in [Y]^{<\omega} \). Then we have the following:

(a) If \( F \in [A]^{<\omega} \) and \( s \in B^F \), then \( q_F^{-1}(\bar{s}) = q_A^{-1}(\pi_{F,A}^{-1}(\bar{s})) \).

(b) \( B_A = \bigcup_{F \in [A]^{<\omega}} \pi_{F,A}^{-1}(B^F) \).

(c) \( B_K = \bigcup_{F \in [Y]^{<\omega}} q_F^{-1}(B^F) \).

**Lemma 4.4.** (a) The family \( B_A \) contains a local base at each point of \( X^A \) in \( \beta(X^A) \), for all \( A \in [Y]^{<\omega} \).

(b) The collection \( B_K \) contains a local base for every point of \( W \in K \).

**Proof.** (a) Observe that the family \( \{ p_{F,A}^{-1}(\bar{s}) : F \in [A]^{<\omega} \text{ and } s \in B^F \} \) is a base of \( X^A \). Since \( \pi_{F,A}^{-1}(\bar{s}) \cap X^A = p_{F,A}^{-1}(\bar{s}) \) for each \( F \in [A]^{<\omega} \) and \( s \in B^F \), from Remark 4.1 we conclude that \( B_A \) contains a local base at each point of \( X^A \) in \( \beta(X^A) \).

(b) Since the family \( \{ p_{F,Y}^{-1}(\bar{s}) : F \in [Y]^{<\omega} \text{ and } s \in B^F \} \) is a base for \( X^Y \) and \( d \) is an embedding, the collection \( \{ d(p_{F,Y}^{-1}(\bar{s})) : F \in [Y]^{<\omega} \text{ and } s \in B^F \} \) is a base for \( d(X^Y) = W \). By using Remark 2.1(c), we can see that \( q_F^{-1}(\bar{s}) \cap W = (q_F \mid_W)^{-1}(\bar{s}) = d(p_{F,Y}^{-1}(\bar{s})) \) for each \( F \in [Y]^{<\omega} \) and \( s \in B^F \). Then we can apply Remark 4.1 to see that the collection \( B_K \) contains a local base for every point of \( W \) in \( K \). \( \square \)

5. INVERSE SYSTEMS AND CONDENSATIONS

Let \( c : S \to X \) be a condensation, that is a continuous and bijective map, from a space \( S \) onto the locally compact non compact space \( X \). We will suppose that the base \( B \) of \( X \), constituted by open subsets with compact closure, satisfies that \( c^{-1}(B) \) is a \( \pi \)-base for the topology of \( S \).

In this section we will use the condensation \( c \) to relate the topological elements defined in Section 2 for the space \( S \) with those defined for the space \( X \). Each of these elements will be escorted by an \( S \) as a subscript or superscript when we move in the \( S \)-world.

The inverse system associated to the space \( S^Y \) and \( K_S \) is the limit of the inverse system \( \mathfrak{M}_S \). We will highlight some natural relations determined by \( c \) between the two topological structures defined by the inverse systems \( \mathfrak{M}_S \) and \( \mathfrak{M}_X \).

For each \( A \subseteq Y \), note that the product map \( c_A := c^A : S^A \to X^A \) is also a condensation. Let \( e_A : \beta(S^A) \to \beta(X^A) \) be the continuous extension of \( c_A \) for every \( A \subseteq Y \). We will denote by \( E_Y \) the product function \( \prod_{G \in [Y]^{<\omega}} E_G : Ps \to P \). Moreover, we set \( E_Y = E_Y \mid K_S \) and \( \zeta_Y = E_Y \mid W_S \).

**Remark 5.1.** (a) The family \( \{ (p_{F^{-1},A}^{-1}(e_F^{-1}(\bar{s})) : F \in [A]^{<\omega} \text{ and } s \in B^F \} \) is a \( \pi \)-base for \( S^A \).

(b) The collection \( \{ d_S((p_{F^{-1},Y}^{-1}(e_F^{-1}(\bar{s}))) : F \in [A]^{<\omega} \text{ and } s \in B^F \} \) is a \( \pi \)-base for \( W_S \).

**Lemma 5.2.** Assume that \( F \subseteq A \subseteq Y \), then we have the following equalities:

(a) \( e_F \circ p_{F,A}^S = p_{F,A} \circ e_A \);

(b) \( e_F \circ p_{F,A}^S = \pi_{F,A} \circ e_A \);

(c) \( \zeta_Y \circ d_S = d \circ c_Y \); and

(d) \( e_A \circ q_A^S = q_A \circ \zeta_Y \) whenever \( A \in [Y]^{<\omega} \).
Proof. (a) For each $f \in S^A$ and $y \in F$ we have that $c_F \circ p_{A,F}^S(f)(y) = (c_F(f|_F))(y) = c(f|_F(y)) = (cA(f))(y) = p_{A,F} \circ c_A(f)(y)$. Hence $c_F \circ p_{A,F}^S = p_{A,F} \circ c_A$.

(b) Note that $e_F \circ s_{A,F} \circ p_{A,F}^S = p_{A,F} \circ e_A = \pi_{F,A} \circ e_A |_{S^A}$. Since $S^A$ is dense in $\beta(S^A)$, we must have that $c_F \circ s_{A,F}^S = \pi_{F,A} \circ e_A$.

(c) Let us observe that $\zeta_Y \circ d_S = E_Y \circ d_S = \prod_{G \in [\gamma]^{\prec \omega}} c_G \circ \Delta_{G \in [\gamma]^{\prec \omega}} = e_{\pi_{F,Y}}(\Delta_{G \in [\gamma]^{\prec \omega}}) = c_G \circ \Delta_{G \in [\gamma]^{\prec \omega}} = d \circ \zeta_Y$.

(d) Given $d_S(f) \in W_S$, it happens that $e_A \circ q_A^S(d_S(f)) = e_A \circ p_{A,Y}^S(f) = (e_A \circ p_{A,Y}^S)(f) = p_{A,Y} \circ c_Y(f) = q_A \circ d \circ c_Y(f) = q_A \circ \zeta_Y \circ d_S(f) = q_A \circ \zeta_Y(d_S(f))$. Since $W_S$ is dense in $K_S$, we must have that $e_A \circ q_A^S = q_A \circ \zeta_Y$.

Diagram 2. All the circles are commutative.

From Lemma 5.2, we deduce that $\zeta_Y = d \circ \zeta_Y \circ d_S^{-1} : W_S \to W$ is a condensation. Since $K_S$ is compact, $W_S$ is dense in $K_S$, $W$ is dense in $K$, and $\zeta_Y \upharpoonright W_S = \zeta_Y$, we must have that $\zeta_Y$ maps $K_S$ continuously onto $K$. We consider the families $B_K^S = \zeta_Y^{-1}(B_K)$ and $B_A^S = \zeta_Y^{-1}(B_A)$, for each $A \in [\gamma]^{\prec \omega}$.

Lemma 5.3. (a) $B_K^S$ is a $\pi$-base for $\beta(S^A)$ and $B_A^S = \bigcup_{F \subseteq [A]^{\prec \omega}} (\pi_{F,A}^{-1}(B_F^S))$ for every $A \in [\gamma]^{\prec \omega}$.

(b) $B_K^S$ is a $\pi$-base for $K_S$ and $B_K^S = \bigcup_{F \subseteq [\gamma]^{\prec \omega}} (q_F^S)^{-1}(B_F^S)$.

Proof. (a) Note that for each $B = \pi_{F,A}^{-1}(\tilde{s}) \in B_A$ we have the equality $c_A^{-1}(B \cap X^A) = c_A^{-1}(\pi_{F,A}^{-1}(\tilde{s}) \cap X^A) = c_A^{-1}(p_{F,A}(\tilde{s})) = (p_{F,A}^S)^{-1}(c_F^{-1}(\tilde{s}))$. Then the family $c_A^{-1}(B_A \cap X^A) = \{(p_{F,A}^S)^{-1}(c_F^{-1}(\tilde{s})) : F \subseteq [A]^{\prec \omega} \}$ is a $\pi$-base for $S^A$. By applying Remark 4.2 we conclude that $B_K^S$ is a $\pi$-base for $\beta(S^A)$.

(b) Observe that for every $B = q_F^{-1}(\tilde{s}) \in B_K$ we have the equality $\zeta_Y^{-1}(B \cap W) = q_F^{-1}(\tilde{s}) \cap W) = \zeta_Y^{-1}(d(p_{F,A}^S(\tilde{s}))) = d_S((p_{F,A}^S(\tilde{s}))) = d_S((p_{F,A}^S)^{-1}(c_F^{-1}(\tilde{s})))$. Then the family $\zeta_Y^{-1}(B_K \cap W) = \{d_S((p_{F,A}^S)^{-1}(c_F^{-1}(\tilde{s}))) : F \subseteq [A]^{\prec \omega} \}$ is a $\pi$-base for $W_S$. By applying Remark 4.2 we conclude that $B_K^S$ is a $\pi$-base for $K_S$.

6. Weak pseudocompactness and $G_\delta$-dense subspaces of $X^Y$

In this section, we prove the main result of our article:
**Theorem 7.1.** Let $C$ be a $G_δ$-dense subspace of $S^Y$ and let $c : S \to X$ be a condensation. Assume that $X$ is not compact and that it has a countable base $B$ such that: $c(B)$ is compact for all $B \in B$, and $c^{-1}(B)$ is a π-base for $S$. Then the space $C$ is not weakly pseudocompact.

**Proof.** Since weak pseudocompactness is productive and the product of $G_δ$-dense subspaces is a $G_δ$-dense subspace of the product, we may assume that $|Y| \geq \omega$.

We will use the constructions from sections 1, 2 and 3. Let $Z = d_S(C)$. By Remark 2.6 it is enough to show that $Z$ is not weakly pseudocompact. Let $L$ be an arbitrary compactification of $Z$. By Remark 2.9, there exists a continuous map $φ : KS \to L$ such that $φ |_Z$ is an embedding. We must show that $φ(Z)$ is not $G_δ$-dense in $L$.

The map $φ$ being closed is a quotient map. According to Lemma 3.2 for each open subset $U$ of $K$ we can fix a nowhere dense subset $N_U$ of $KS$ such that $U \cup N_U$ is closed and $φ$-saturated in $KS$. For each $A \in [Y]^{≤ω}$ let $N_A^S = \{NU : U \in (q_A^S)^{-1}(B_δ^S)\}$. Note that $N_A^S = \bigcup_{F ∈ [A]^{≤ω}} N_F^S$ for all $A \in [Y]^{≤ω}$.

Because of Lemmas 3.3 and 5.3 we can find an $A \in [Y]^{≤ω}$ such that $q_A^S(N_A^S)$ is a family of nowhere dense subsets of $β(S^A)$. By applying Remark 3.4 and Lemma 5.3 we deduce that $e_A(q_A^S(N_A^S))$ is a family of nowhere dense subsets of $β(X^A)$. Since $e_A(q_A^S(N_A^S))$ is countable and $β(X^A)$ has the Baire property, the set $N_A = \bigcup e_A(q_A^S(N_A^S))$ is nowhere dense in $β(X^A)$.

Let $D_A = β(X^A) \setminus X^A$. Note that the space $X^A$ is nowhere locally compact because $A$ is infinite and $X$ is not compact. As a consequence, the set $D_A$ is dense in $β(X^A)$. It follows that $D_A \setminus N_A \neq ∅$. Fix a map $f_A \in D_A \setminus N_A$. By Lemma 4.4, for each $g \in X^A$ we can find $B_g \in B_A$ such that $g ∈ B_g$ and $f_A \notin B_g$.

The space $X^A$ being second countable has the Lindelöf property. Since the family $\{B_g : g \in X^A\}$ covers $X^A$, it contains a countable subfamily $C_A$ which covers $X^A$. Note that $C_A ⊆ B_A$ and $f_A \notin \bigcup C_A$.

We are ready to show that $φ(Z)$ is not $G_δ$-dense in $L$. Observe that $G = \{β(S^Y) \setminus (U \cup N_U) : U ∈ (q_A^S)^{-1}(e_A^{-1}(C_A))\}$ is a countable family of open $φ$-saturated subsets of $KS$ and, as a consequence, $φ(G)$ is a countable family of open subsets of $L$.

**Claim.** The $G_δ$-set $∩ φ(G)$ is nonempty and misses $φ(Z)$.

**Proof of the claim.** For the proof of the first part of the Claim, fix $f ∈ (q_A^S)^{-1}(e_A^{-1}(f_A))$. Pick $U ∈ (q_A^S)^{-1}(e_A^{-1}(C_A))$ arbitrarily and choose $B ∈ C_A$ such that $U = (q_A^S)^{-1}(e_A^{-1}(B))$. From $B ∈ C_A ⊆ B_A$ we deduce that $f_A \notin B = e_A(q_A^S(U))$. Besides $U ∈ (q_A^S)^{-1}(e_A^{-1}(B_A)) = (q_A^S)^{-1}(B_δ^S)$ and in this way $N_U ∈ N_A^S$. We know that $f_A \notin N_A$ and in particular $f_A \notin e_A(q_A^S(N_U))$. From these facts we deduce that $f_A \notin e_A(q_A^S(U \cup N_U))$. Hence $f ∈ β(S^Y) \setminus (U \cup N_U)$. Since $U$ is arbitrary, we conclude that $f ∈ ∩ G$. Therefore $φ(f) ∈ ∩ φ(G)$.

For the proof of the second part of the Claim, note that $(∩ G) ∩ ∪ (q_A^S)^{-1}(e_A^{-1}(C_A)) = ∅$. Then $(∩ G) \cap e_A^{-1}(∪ C_A) = ∅$. The fact that $C_A$ is a cover of $X^A$ implies that $S^A ⊆ e_A^{-1}(∪ C_A)$. Hence $(∩ G) \cap S^A = ∅$, that is, $(∩ G) ∩ (q_A^S)^{-1}(S^A) = ∅$. In particular $(∩ G) ∩ Z = ∅$. Therefore $(∩ φ(G)) \cap φ(Z) = ∅$. □

7. SOME CONSEQUENCES OF THEOREM 6.1

Since the identity function is a condensation and every base is a π-base, we have:

**Theorem 7.1.** Let $C$ be a $G_δ$-dense subspace of $X^Y$. If $X$ is a non compact locally compact second countable space, then the space $C$ is not weakly pseudocompact.

Our first corollary includes Theorem 1.3.

**Corollary 7.2.** Let $G$ be a metrizable separable locally compact non compact topological group and let $κ$ be a cardinal number. Then, the space $C_κ(X, G^κ)$ is never weakly pseudocompact.

**Proof.** Assume that there exists a space $X$ such that $C_κ(X, G^κ)$ is weakly pseudocompact. The space $C_κ(X, G^κ)$ is homeomorphic to $C_κ(⊕_{ξ < κ} X_ξ, G)$ where each $X_ξ = X$. It was proved in [9, Corollary 6.5] that this implies that $C_κ(⊕_{ξ < κ} X_ξ, G)$ is $G_δ$-dense in $G^{⊕_{ξ < κ} X_ξ}$. Because of Theorem 6.1, the space $C_κ(X, G^κ)$ is not weakly pseudocompact; a contradiction. □

**Corollary 7.3.** For every Tychonoff space $X$ and every cardinal number $κ$, the space $C_κ(X, ℝ^κ)$ is not weakly pseudocompact.

**Corollary 7.4.** For every zero-dimensional space $X$ and every cardinal number $κ$, the space $C_κ(X, Z^κ)$ is not weakly pseudocompact.

Since the space of irrational numbers with its Euclidean topology $P$ is homeomorphic to $Z^ω$, we obtain:
Corollary 7.5. For every zero-dimensional space $X$ and every cardinal number $\kappa$, the space $C_p(X, \mathbb{P}^\kappa)$ is not weakly pseudocompact.

Corollary 7.6. The spaces $\mathbb{R}^\kappa$, $\mathbb{N}^\kappa$ and $\mathbb{P}^\kappa$ are not weakly pseudocompact, for any cardinal $\kappa$.

The previous result gives us examples of non Lindelöf $\mathbb{R}$-factorizable Sánchez-Okunev countably compact Oxtoby countably compact topological groups with weight $\kappa \cdot \omega$, for any cardinal $\kappa$, which are not weakly pseudocompact. The space $\mathbb{R}^\kappa$ is even Telgársky countably compact (see definitions in Section 3 in [4]; see also Theorem 4.6 and Question 11.5 in the same article).

For a filter $\mathcal{F}$ on $\kappa$, a topological space $X$, and $o \in X$, we consider the space $\Sigma_{\mathcal{F}, o} X^\kappa := \{x \in X^\kappa : \{\xi < \kappa : x(\xi) = o\} \in \mathcal{F}\}$ which is the so called $\Sigma_{\mathcal{F}}$-product on $X^\kappa$ based on $o$. If $\mathcal{F}$ is $\omega$-complete (that is, if for every sequence $F_0, \ldots, F_n, \ldots$ of elements in $\mathcal{F}$, $\bigcap_{n<\omega} F_n \in \mathcal{F}$), then $\Sigma_{\mathcal{F}} X^\kappa$ is $G_\delta$-dense in $X^\kappa$.

Corollary 7.7. Let $\mathcal{F}$ be an $\omega$-complete filter on $\kappa$. Let $G$ be a metrizable separable locally compact non compact topological group. Then $\Sigma_{\mathcal{F}, o} G^\kappa$ is not weakly pseudocompact.

Let $\kappa$ be a cardinal number such that $\text{cof}(\kappa) > \omega$. Let $\mathcal{F}_\omega$ be equal to the filter on $\kappa$ of all the subsets of $\kappa$ with countable complement. Then $\mathcal{F}_\omega$ is $\omega$-complete. So:

Corollary 7.8. Let $\kappa$ be a cardinal number such that $\text{cof}(\kappa) > \omega$. Let $G$ be a metrizable separable locally compact non compact topological group. Then $\Sigma_{\mathcal{F}_\omega, o} G^\kappa$ is not weakly pseudocompact. In particular, the classic $\Sigma$-products $\Sigma \mathbb{R}^\kappa$ and $\Sigma \mathbb{N}^\kappa$ are not weakly pseudocompact.

Since the Sorgenfrey line $\mathbb{S}$ can be condensed on the real line and the collection of open intervals are a $\pi$-base in $\mathbb{S}$, we conclude from Theorem 6.1 the following:

Corollary 7.9. For every cardinal number $\kappa$, $\mathbb{S}^\kappa$ and $\Sigma \mathbb{S}^\kappa$ are not weakly pseudocompact.

We finish our list of corollaries by mentioning the Niemytzki Plane $\mathbb{M}$ (see [7]). Observe that $\mathbb{M}$ is a Baire non Lindelöf space; and in addition $\mathbb{M}$ can be condensed into the Euclidean semiplane. So, we obtain:

Corollary 7.10. For every cardinal number $\kappa$, the spaces $\mathbb{M}^\kappa$ and $\Sigma \mathbb{M}^\kappa$ are not weakly pseudocompact.

In order to present our last remark we need the following definition. Let $Y$ be a topological space. A space $X$ is $u_\gamma$-discrete if every countable subset $N$ of $X$ satisfies: (i) $N$ is closed, and (ii) every continuous function $f : N \to Y$ can be extended to a continuous function $F : X \to Y$.

Remark 7.11. None of the hypotheses in Theorem 6.1 are essential as Corollary 7.5 shows when $\kappa > \gamma$. On the other hand, separability is essential in Corollary 7.2. Indeed, in [2] it was proved that for a zero-dimensional $u_\kappa$-discrete space $X$ and for the discrete space of cardinality $\kappa > \omega$, $D(\kappa)$, $C_p(X, D(\kappa))$ is a weakly pseudocompact topological group (and non locally pseudocompact if $X$ is infinite). (A more general result is: for every weakly pseudocompact space $Y$ and every $u_\gamma$-discrete space $X$, $C_p(X, Y)$ is weakly pseudocompact).

8. Open problems

Problem 8.1. Is it true that $C_p(X, Y)$ is not weakly pseudocompact when $Y$ is metrizable separable locally compact non compact? That is, is the version of Corollary 7.2 in which we do not ask the group topology structure on $G$ true?

We know that for every $u_\gamma$-discrete space $X$ (see definition before Remark 7.11), $C_p(X, Y)$ is weakly pseudocompact if $Y$ is weakly pseudocompact (see [3]); then the following question arises:

Problem 8.2. Let $X$ be a $u_\gamma$-discrete space. Must $Y$ be weakly pseudocompact if $C_p(X, Y)$ is weakly pseudocompact?

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