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On discretely generated box products

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ABSTRACT

A topological space X is called *discretely generated* if for any $A \subseteq X$ and $x \in \overline{A}$ there exists a discrete set $D \subseteq A$ such that $x \in \overline{D}$. We solve the Problems 3.19 and 3.3 in [2]. Problem 3.19: Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines when $\xi \in \beta\omega \setminus \omega$? For any $\xi \in \beta\omega \setminus \omega$, we answer negatively. Problem 3.3: Is any box product of first countable spaces discretely generated? We answer positively by assuming that the spaces are regular.

1. Introduction

Many results concerning to discretely generated spaces have been shown in [1] and [2]. Theorem 2.6 [2] by V. Tkachuk and R. Wilson says that if X_t is a monotonically normal space, then the box product $\square_{t\in T} X_t$ is discretely generated. Hence, the spaces $\square\mathbb{R}^\kappa$, $\square(\omega+1)^\kappa$ and $\square(\{\xi\}\cup\omega)^\kappa$ are discretely generated, for any cardinal κ.

Let $\mathcal{V}$ be the countable regular maximal space due to Eric van Douwen [3]. It was shown in [1] that $\mathcal{V}$ is not discretely generated. Since $\square\mathbb{R}^\kappa$ is discretely generated and this property is hereditary, there is no embedding from $\mathcal{V}$ to $\square\mathbb{R}^\kappa$. The authors of [2] then wondered if there were more countable regular spaces that do not embed into a box product of real lines, that is the motivation of Problem 3.19 in [2]. We generalize their Example 2.10, part b).

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A space X is called *monotonically normal* if for every $U \in \tau(X)$ and $x \in U$ there is a set $O(x,U) \in \tau(x,X)$ such that $O(x,U) \cap O(y,V) = \emptyset$ implies $x \in V$ or $y \in U$. Of course, being monotonically normal implies normality. Every metric space is monotonically normal. However, there is no relation between being first countable and monotonically normal. For example, $\{\xi\} \cup \omega$ is a monotonically normal non first countable space. On the other hand, it is well known that the square of the Sorgenfrey line $\mathbb{R}_l^2$ is a regular first countable non normal space, and thus, non monotonically normal. However, the space $\square\mathbb{R}_l^\omega$ is discretely generated by our result.

2. Strategy, notation and terminology

We use standard terminology and follow Engelking [4]. All spaces we consider are assumed to be Hausdorff. If X is a space then $\tau(X)$ is its topology. If X_t is a topological space for every $t \in T$, then the *box product* $\square_{t\in T} X_t$ is the set-theoretic product $\prod_{t\in T} X_t$ with the topology generated by the family $\{\prod_{t\in T} U_t : U_t \in \tau(X_t)\}$. The set of natural numbers is denoted by ω and we use the symbol $\mathbb{R}$ for the real line with its usual topology.

A space X is *discretely generated at a point* $x \in X$ if for any $A \subseteq X$ with $x \in \overline{A}$ there exists a discrete set $D \subseteq A$ such that $x \in \overline{D}$. The space X is *discretely generated* if it is discretely generated at every point $x \in X$.

Let X be a set, $A \subseteq X^\kappa$, κ a cardinal, $S \subseteq \kappa$ and $b \in X^\kappa$. We denote the *support of* $a \in X^\kappa$ *respect to* b by $supp_b(a) = \{\alpha \in \kappa : a(\alpha) \neq b(\alpha)\}$. The *restriction of* a *to* S is the element $a \restriction S \in X^S$ defined as $(a \restriction S)(s) = a(s)$, as well as $A_{S,b} = \{a \in A : supp_b(a) = S\}$ and $A \restriction S = \{a \restriction S \in X^S : a \in A\}$. We denote by $\overrightarrow{\omega}$ the element in $\square(\omega+1)^\omega$ such that for every $n \in \omega$, $\overrightarrow{\omega}(n) = \omega$. When we talk about the "support" in $\square(\omega+1)^\omega$, we use $supp(a)$ instead of $supp_{\overrightarrow{\omega}}(a)$ and A_S instead of $A_{S,\overrightarrow{\omega}}$.

Also, given a function $h \in \omega^\omega$ and an element $a \in \square(\omega+1)^\omega$, we define the *neighborhood of* a *by* h to be the set of the form

$$N_h(a) = \square\{\{a(n)\} : n \in supp(a)\} \times \square\{(h(n), \omega] : n \in \omega \setminus supp(a)\}.$$

Finally, we recall the following definitions on ω^ω: For $f, g \in \omega^\omega$, define $f \leq^* g$ iff $\exists n \in \omega\ \forall m \geq n\ (f(m) \leq g(m))$. A family $\mathcal{F} \subseteq \omega^\omega$ is $\leq^*$-*bounded* if $\exists g \in \omega^\omega\ \forall f \in \mathcal{F}\ (f \leq^* g)$. A family $\mathcal{F} \subseteq \omega^\omega$ is $\leq^*$-*dominant* if $\forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ (g \leq^* f)$.

- $\mathfrak{b} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is not } \leq^*\text{-bounded}\}$
- $\mathfrak{d} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is } \leq^*\text{-dominant}\}$

3. Facts and some definitions

Let $\beta\omega$ denote the Stone–Čech compactification of ω. If $\xi \in \beta\omega \setminus \omega$, then $\{\xi\} \cup \omega$ inherits the subspace topology of $\beta\omega$.

Remark 1. Let $\xi \in \beta\omega \setminus \omega$, then we have the following for the space $\{\xi\} \cup \omega$:

1. $U \in \xi$ if and only if $\xi \in \overline{U}$.
2. If $\xi \in \overline{U} \cap \overline{V}$, then $U \cap V \neq \emptyset$.

Lemma 2. *If a set* $A \subseteq \square(\omega+1)^\omega$ *satisfies* $\forall a \in A\ (|supp(a)| = \omega)$ *and has size less than* $\mathfrak{b}$, *then* $\overrightarrow{\omega} \notin \overline{A}$.

Proof. Let

$$A_0 = \{g \in \omega^\omega : \exists f \in A \ ((g \restriction supp(f)) = (f \restriction supp(f)) \ \wedge \ (g \restriction \omega \setminus supp(f)) = 0)\}.$$

Note that $|A_0| = |A| < \mathfrak{b}$. Since $A_0 \subseteq \omega^\omega$ is $\leq^*$-bounded, there exists $h \in \omega^\omega$ such that $\forall g \in A_0 \ (g \leq^* h)$. Now is immediate that $N_h(\overrightarrow{\omega}) \cap A = \emptyset$. ■

Lemma 2 will help us to isolate a set of those elements with infinity support as long as the size of the set be less than $\mathfrak{b}$. We can also isolate sets consisting of elements with finite support under certain conditions which we will establish in the next lemma.

Lemma 3. *Suppose that $A \subseteq \Box(\omega+1)^\omega$ and $\forall a \in A \ (|supp(a)| < \omega)$. Let $\mathcal{F} = \{F \in [\omega]^{<\omega} : |A_F| < \omega\}$. Then exists $h \in \omega^\omega$ such that $N_h(\overrightarrow{\omega}) \cap (\bigcup_{F \in \mathcal{F}} A_F) = \emptyset$.*

Proof. Consider an enumeration for $\bigcup \mathcal{F} = \{n_k : k \in \omega\}$. Observe that for each $k \in \omega$, the set $B_k = \{F \in \mathcal{F} : F \subseteq \{n_0, \ldots, n_k\}\}$ is finite and so is $\bigcup_{F \in B_k} A_F$. Hence, for each $n_k \in \bigcup \mathcal{F}$, choose $h(n_k) > max\{\pi_{n_k}[\bigcup_{F \in B_k} A_F]\}$. Now, if $n \notin \bigcup \mathcal{F}$ let $h(n) = 0$. Then $h \in \omega^\omega$ works as required. ■

The lemmas above try to “ward off” from $\overrightarrow{\omega}$ certain kind of elements in a set with a small size. However, the role of the support type in the elements is important even for small sets. The next example provides a countable set whose elements have finite support and it has $\overrightarrow{\omega}$ in its closure.

Example 4. There exists a countable set $A \subseteq \Box(\omega+1)^\omega$ of elements with finite support such that $\forall F \in [\omega]^{<\omega}(\overrightarrow{\omega} \notin \overline{A_F})$, but $\overrightarrow{\omega} \in \overline{A}$.

Proof. The condition $\forall F \in [\omega]^{<\omega}(\overrightarrow{\omega} \notin \overline{A_F})$ only avoids the trivial case when $\overrightarrow{\omega} \in \overline{A_F}$, for some $F \in [\omega]^{<\omega}$.

For every $n \in \omega$, consider the set

$$A_n = \{a \in \Box(\omega+1)^\omega : a(0) = n \wedge supp(a) = n\}.$$

We will see that $A = \bigcup_{n \in \omega} A_n$ works. It is clear that each element of A has finite support. Now, note that $A_F = A_n$ if and only if $F = n$, for every $n \in \omega$. Moreover, $\overrightarrow{\omega} \notin \overline{A_n}$ because $a(0) = n$, for $a \in A_n$.

Claim 4.1. $\overrightarrow{\omega} \in \overline{A}$.

Consider a function $h \in \omega^\omega$. Let $k \in \omega$ such that $k > h(0)$. So, there is $a \in A_k$ such that $\forall i \leq k((a \restriction k)(i) > h(i))$ and $\forall i > k((a \restriction k)(i) = \omega)$. That is, $a \in N_h(\overrightarrow{\omega})$. ■

Let X be a space. Given $A \subseteq X$ and $x \in A' = \overline{A} \setminus A$, we say that X has the *property* $\mathcal{P}$ at (x, A) if $\exists B, C \subseteq A \ (B \cap C = \emptyset \wedge x \in B' \cap C')$. Also, X has the *property* $\mathcal{P}^+$ if $\forall A \subseteq X \ \exists B, C \subseteq A \, (B \cap C = \emptyset \wedge A' = B' = C')$. It is clear that property $\mathcal{P}^+$ is stronger than property $\mathcal{P}$ and both of them are topological properties. It is worth to mention that the space $\{\xi\} \cup \omega$ does not satisfy the property $\mathcal{P}$ at any (ξ, U), with $U \in \xi$. This implies the fact that if the space $\{\xi\} \cup \omega$ embeds in X via φ, then X does not satisfy the property $\mathcal{P}$ at $(\varphi(\xi), \varphi[\omega])$. In the next lemma we will use this properties.

Remark 5. Any second countable space has the property $\mathcal{P}^+$.

Lemma 6. *Suppose $A \subseteq \Box(\omega+1)^\omega$ satisfies that $\forall a \in A \ (|supp(a)| < \omega)$. If $\overrightarrow{\omega} \in A'$, then the property $\mathcal{P}$ is satisfied at $(\overrightarrow{\omega}, A)$. Moreover, it is satisfied $\mathcal{P}$ at (x, A), for any $x \in A'$.*

Proof. By Lemma 3, we may assume without loss of generality that $\forall F \in [\omega]^{<\omega}$, A_F is infinite. So, if F and F' are two "different" finite sets of naturals, then $A_F \cap A_{F'} = \emptyset$.

Case 1: If $\overrightarrow{\omega} \in \overline{A_F}$ for some finite set $F \subseteq \omega$, then there is a sequence in A_F converging to $\overrightarrow{\omega}$, because $(\omega+1)^F$ is a metric space. If B consists of the even terms of that sequence and C of the odd terms, we get the required.

Case 2: For each finite set F, $\overrightarrow{\omega} \notin \overline{A_F}$. Look at A_F as a subset of ω^F and note that A_F is an infinite discrete set. Moreover, $(\square(\omega+1)^\omega)_F$ is homeomorphic to $(\omega+1)^F$ and so it is a compact metric space. Then A'_F is not empty and by the remark above, it is possible to get $B_F, C_F \subseteq A_F$ such that these sets are pairwise disjoint and $A'_F = B'_F = C'_F$.

Since $A = \bigsqcup_{F\in[\omega]^{<\omega}} A_F$, it follows that $B := \bigsqcup_{F\in[\omega]^{<\omega}} B_F$ and $C := \bigsqcup_{F\in[\omega]^{<\omega}} C_F$ are subsets of A. Furthermore, $B \cap C = \emptyset$.

Claim 6.1. $\overrightarrow{\omega} \in B' \cap C'$.

Consider $h \in \omega^\omega$. Apply Lemma 3 to get a finite set F of naturals such that $N_h(\overrightarrow{\omega}) \cap A_F$ is infinite. As before, $N_h(\overrightarrow{\omega}) \cap A_F \subseteq (\square(\omega+1)^\omega)_F \simeq (\omega+1)^F$, and so, $(N_h(\overrightarrow{\omega}) \cap A_F)' \neq \emptyset$. By construction,

$$(N_h(\overrightarrow{\omega}) \cap A_F)' = (N_h(\overrightarrow{\omega}) \cap B_F)' = (N_h(\overrightarrow{\omega}) \cap C_F)'.$$

Necessarily there exists $b \in B_F \subseteq B$ and $c \in C_F \subseteq C$ such that $b, c \in N_h(\overrightarrow{\omega})$.

Finally, suppose that $x \in A'$ and let $S = supp(x)$. If S is coinfinite, we can assume that $A \restriction S = \{x \restriction S\}$ and apply the case when the space $\square(\omega+1)^{\omega\setminus S}$ has the property $\mathcal{P}$ at $(x \restriction (\omega \setminus S), A \restriction (\omega \setminus S))$, because $x \restriction (\omega\setminus S) = \overrightarrow{\omega} \restriction (\omega\setminus S)$. Now, if S is cofinite, then $(\square(\omega+1)^\omega)_{\omega\setminus S} \simeq (\omega+1)^{\omega\setminus S}$, which is second countable. By remark above the property $\mathcal{P}$ is satisfied. ■

The next step is to involve the box product of real lines since the original question refers to it. The lemmas proved before must be true also for the countable box product of real lines. In the following, we will relate in a "natural way" both spaces $\square(\omega+1)^\omega$ and $\square\mathbb{R}^\omega$.

Consider $p \in \square\mathbb{R}^\omega$. For each $n \in \omega$, define $W^n_k = (p(n) - \frac{1}{k}, p(n) + \frac{1}{k})$ and $W^n_0 = \mathbb{R}$. Note that $\{\square_{n\in\omega} W^n_{h(n)} : h \in \omega^\omega\}$ is a local basis for p. For every $n \in \omega$, define $f_n : \mathbb{R} \to \omega+1$ as

$$f_n(a) = \begin{cases} k, & \textit{if } a \in W^n_k \setminus W^n_{k+1}; \\ \omega, & \textit{if } a = p(n). \end{cases}$$

Let $\Phi_p : \square\mathbb{R}^\omega \to \square(\omega+1)^\omega$ be the diagonalization map for p given by $\Phi_p(a) = \langle f_n(a(n)) : n \in \omega\rangle$. It is clear that $\Phi_p(p) = \overrightarrow{\omega}$.

Note that $\Phi_p^{-1}[\square(\omega+1)^\omega]$ induce a relation of equivalence (or *partition induced by* p) over the space $\square\mathbb{R}^\omega$. That is, for $a, b \in \square\mathbb{R}^\omega$, we have that $a \sim_p b$ if and only if

$$\forall n \in \omega \ \ (f_n(a(n)) = f_n(b(n))).$$

Given $A \subseteq \square\mathbb{R}^\omega$, let $R_A \subseteq A$ be a set of $\sim_p$-representative elements for the partition induced by p. We have the following properties.

Remark 7.

1. If $A \subseteq \square\mathbb{R}^\omega$ and $p \in \overline{A}$, then $p \in \overline{R_A}$.
2. If $A \subseteq \square\mathbb{R}^\omega$ and R_A is a set of $\sim_p$-representative elements of A, then $\Phi_p \restriction R_A$ is a one-to-one mapping.

3. The mapping Φ_p induces an assignment between neighborhood systems for p and for $\overrightarrow{\omega}$. That is, for every $h \in \omega^\omega$, it follows that $\Phi_p[\Box_{n\in\omega} W^n_{h(n)}] = N_h(\overrightarrow{\omega})$ and also $\Box_{n\in\omega} W^n_{h(n)} = \Phi_p^{-1}[N_h(\overrightarrow{\omega})]$.

Note that Lemmas 2 and 3 are purely combinatorial. Once we have a subset $A \subseteq \Box\mathbb{R}^\omega$, we can always consider a set R_A of $\sim_p$-representative elements of A to work with it. Hence, by Remark 7, the same combinatorial properties holds for the space $\Box\mathbb{R}^\omega$ or rather $R_{\Box\mathbb{R}^\omega}$. So, Lemmas 2 and 3 are also true if we change $\Box(\omega+1)^\omega$ for $\Box\mathbb{R}^\omega$ and $\overrightarrow{\omega}$ for a fixed $p \in \Box\mathbb{R}^\omega$. In the next we answer partially the original question, but this is the key step to generalize the result.

Theorem 8. *There is no embedding* $\varphi : \{\xi\} \cup \omega \to \Box\mathbb{R}^\omega$.

Proof. Suppose there is such embedding φ. Let $A = \varphi[\omega] \subseteq \Box\mathbb{R}^\omega$, $p = \varphi(\xi)$ and consider the partition induced by p on $\Box\mathbb{R}^\omega$. Since A is countable, by Lemmas 2 and 3, we may assume that for every $a \in A$ the support $supp_p(a)$ is finite. Also, suppose that A consists of $\sim_p$-representative elements.

Claim 8.1. $\overrightarrow{\omega} \in \overline{\Phi_p[A]}$.

In fact, given $h \in \omega^\omega$, let $W = \Box_{n\in\omega} W^n_{h(n)}$. Then, there is $a \in W \cap A$. So, $\Phi_p(a) \in N_h(\overrightarrow{\omega})$.

Now, apply Lemma 6 to get $B_0, C_0 \subseteq \Phi_p[A]$ such that $B_0 \cap C_0 = \emptyset$ and $\overrightarrow{\omega} \in \overline{B_0} \cap \overline{C_0}$. Let $B = \Phi_p^{-1}[B_0] \cap A$ and $C = \Phi_p^{-1}[C_0] \cap A$. It is clear that $B \cap C = \emptyset$.

Claim 8.2. $p \in \overline{B} \cap \overline{C}$.

Again, let $W = \Box_{n\in\omega} W^n_{h(n)}$ be a neighborhood of p. There are $b_0 \in N_h(\overrightarrow{\omega}) \cap B_0$ and $c_0 \in N_h(\overrightarrow{\omega}) \cap C_0$. Finally, there are $b, c \in \Phi_p^{-1}[\{b_0, c_0\}]$ such that $b \in B \cap W$ and $c \in C \cap W$.

In other words, property $\mathcal{P}$ is satisfied at (p, A). Now, consider the subsets $U = \varphi^{-1}[B]$ and $V = \varphi^{-1}[C]$ of $\{\xi\} \cup \omega$. These sets satisfied that $U \cap V = \emptyset$ and $\xi \in \overline{U} \cap \overline{V}$, but this contradicts Remark 1. ∎

Corollary 9. *For any cardinal* κ*, there is no embedding from* $\{\xi\} \cup \omega$ *to* $\Box\mathbb{R}^\kappa$.

Proof. Suppose again that $\varphi : \{\xi\} \cup \omega \to \Box\mathbb{R}^\kappa$ is an embedding. Let $p = \varphi(\xi)$ and $A = \varphi[\omega]$. We partition $A = A^* \sqcup A_\infty$, where

$$A^* = \{a \in A : |supp_p(a)| < \omega\} \text{ and } A_\infty = \{a \in A : |supp_p(a)| \geq \omega\}.$$

For every $a \in A_\infty$, consider any fixed countable set $S_a \subseteq supp_p(a)$. Since A_∞ is countable, so is $S = \bigcup_{a\in A_\infty} S_a \subseteq \kappa$. Then, note that $A_\infty \restriction S \subseteq \Box\mathbb{R}^S$ is countable. We can apply the same method of the proof in Lemma 2 to find $h \in \omega^S$ such that $(A_\infty \restriction S) \cap \Box_{s\in S} W^s_{h(s)} = \emptyset$. Now, consider the function $H \in \omega^\kappa$ defined as $H(s) = h(s)$ if $s \in S$, and $H(s) = 0$ otherwise. It is clear now that $A_\infty \cap \Box_{s\in\kappa} W^s_{H(s)} = \emptyset$.

Thus, without loss of generality, assume that every element in A has finite support respect of p. Note that $S = \bigcup_{a\in A^*} supp_p(a)$ is countable and $A^* \cong A^* \restriction_S \subseteq \Box\mathbb{R}^S \cong \Box\mathbb{R}^\omega$. We already reflected the problem to the partial case of ω factors. The proof is done by following the same method as the proof of Theorem 8. ∎

Note that the method to solve Problem 3.19 uses only combinatorial and topological properties of $\Box(\omega+1)^\omega$ as well as some trivial properties of ultrafilters. Therefore, Corollary 9 can be generalized as the following sentence: If X is a topological space with $|X| < \mathfrak{b}$ and $p \in \beta X \setminus X$ an ultrafilter, then there is no embedding from $\{p\} \cup X$ to $\Box\mathbb{R}^\kappa$, for any cardinal κ.

In [2], Example 2.10 part b) they mention that the space $\{p\} \cup \mathbb{Q}$ is not discretely generated when p is a remote point, and so, it can not be embedded into a box product of real lines. The remark above generalizes the fact for all ultrafilters $p \in \beta\mathbb{Q} \setminus \mathbb{Q}$.

4. Box products of regular first countable spaces are discretely generated

The diagonalization map Φ was defined over $\Box\mathbb{R}^\omega$. We need the metric of $\mathbb{R}$ to define it, but we can weaken the definition of Φ only to the property of being first countable. Suppose $\{X_t : t \in T\}$ is a family of regular first countable spaces and fix $p \in \Box_{t\in T} X_t$. For each $t \in T$, let $\{W_k^t : k \in \omega\}$ be a countable base for $p(t)$ such that $W_0^t = X_t$ and $\overline{W_{k+1}^t} \subseteq W_k^t$, for each $k \in \omega$. For every $t \in T$, define $f_t : X_t \to \omega + 1$ as

$$f_t(x) = \begin{cases} k, & \textit{if } x \in W_k^t \setminus W_{k+1}^t; \\ \omega, & \textit{if } x = p(t). \end{cases}$$

Just like before, $\vec{\omega} \in \Box(\omega+1)^T$ is the constant element equal ω in each coordinate. Then, $\Phi_p : \Box_{t\in T} X_t \to \Box(\omega+1)^T$ and $\Phi_p(p) = \vec{\omega}$. Also, we will use the relation of equivalence $\sim_p$ defined in the previous section with the obvious generalization for the space $\Box_{t\in T} X_t$ ($a \sim_p b$ iff $\forall t \in T\ f_t(a(t)) = f_t(b(t))$). From now, we will write N_h for $N_h(\vec{\omega})$, the neighborhoods of $\vec{\omega}$.

Definition 10. If κ is a cardinal, we define $\leq^*$ on ω^κ such that for $f, g \in \omega^\kappa$, then $f \leq^* g$ if and only if $\{\alpha \in \kappa : f(\alpha) > g(\alpha)\}$ is finite. We say that $\mathcal{D} \subseteq \omega^\kappa$ is a $\leq^*$-dominant family if $\forall f \in \omega^\kappa\ \exists g \in \mathcal{D}\ (f \leq^* g)$. Recall that

$$\mathfrak{d}(\kappa) = min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\kappa \text{ is a } \leq^* \text{-dominant family}\}.$$

If $\mathcal{D} = \{g_\alpha \in \omega^\kappa : \alpha < \mathfrak{d}(\kappa)\}$ is a $\leq^*$-dominant family, then $\mathcal{D}^* = \{h \in \omega^\kappa : |\{\beta \in \kappa : h(\beta) \neq g_\alpha(\beta)\}| < \omega\ \wedge\ g_\alpha \in \mathcal{D}\}$ is a $\leq$-dominant family, with $|\mathcal{D}^*| = \kappa\mathfrak{d}(\kappa)$ and $\{N_h : h \in \mathcal{D}^*\}$ is a local basis of $\vec{\omega}$. By Lemma 2.1 in [5], $\mathfrak{d}(\kappa) = \kappa \cdot \mathfrak{d}(\kappa)$. Thus, enumerating $\mathcal{D}^* = \{h_\alpha \in \omega^\kappa : \alpha \in \mathfrak{d}(\kappa)\}$ we have that $\{N_{h_\alpha} : \alpha < \mathfrak{d}(\kappa)\}$ is a local basis of $\vec{\omega}$.

Theorem 11. *Suppose $\{X_t : t \in T\}$ is a family of regular first countable spaces. Then $\Box_{t\in T} X_t$ is discretely generated.*

Proof. Let $A \subseteq \Box_{t\in T} X_t$ and $p \in \overline{A} \setminus A$. We may assume that A consists only of $\sim_p$-representative elements. Let $E = \Phi_p[A]$. Note that $\Phi_p(p) = \vec{\omega} \in \overline{E}$. As $\Box(\omega+1)^T$ is discretely generated, there is $D \subseteq E$ discrete such that $\vec{\omega} \in \overline{D}$. Consider a local basis $\{N_{h_\alpha} : \alpha < \mathfrak{d}(\kappa)\}$ of $\vec{\omega}$. Recursively construct a set $\{d_\alpha : \alpha < \mathfrak{d}(\kappa)\}$ as follows. Take $d_0 \in N_{h_0} \cap D$.

- Successor case: Suppose constructed $D_\alpha = \{d_\beta : \beta \leq \alpha\}$ and let $\gamma = \alpha + 1$. If $\vec{\omega} \in \overline{D_\alpha}$, we are done. If not, there is $e_\gamma \in \omega^\kappa$ such that $N_{e_\gamma} \cap D_\alpha = \emptyset$. Since $\vec{\omega} \in \overline{D}$, there is $d_\gamma \in N_{(e_\gamma)+2} \cap N_{h_\gamma} \cap D$.
- Limit case: Suppose constructed $D_{<\alpha} = \{d_\beta : \beta < \alpha\}$. If $\vec{\omega} \in \overline{D_{<\alpha}}$, we are done. If not, there is $e_\alpha \in \omega^\kappa$ such that $N_{e_\alpha} \cap D_{<\alpha} = \emptyset$. Since $\vec{\omega} \in \overline{D}$, there is $d_\alpha \in N_{e_\alpha+2} \cap N_{h_\alpha} \cap D$.

The reason to add 2 to the functions e_α is to choose the elements d_α sufficiently far away from each other just to be sure their preimages are still enough apart from each other, that will allow us to find a discrete subset of A.

We may assume that the process ends until $\vec{\omega} \in \overline{D_{\mathfrak{d}(\kappa)}}$. Note that $D_{\mathfrak{d}(\kappa)}$ is discrete because it is contained in D. Now, let $G = \Phi_p^{-1}[D_{\mathfrak{d}(\kappa)}] \cap A$. From the construction of $D_{\mathfrak{d}(\kappa)}$, we have the following property for $\beta < \alpha < \mathfrak{d}(\kappa)$:

- $\exists t \in supp_{\vec{\omega}}(d_\beta)\ \ (d_\beta(t) + 2 < d_\alpha(t))$.

Claim 11.1. The set $G \subseteq A$ is discrete and $p \in \overline{G}$.

By the Remark 7, it is clear that $p \in \overline{G}$. To see that G is discrete, fix $g \in G$. There is $\alpha < \mathfrak{d}(\kappa)$ such that $d_\alpha = \Phi_p(g)$. For every $t \in supp_p(g)$, there is a unique $k_t \in \omega$ such that $g(t) \in W^t_{k_t} \setminus W^t_{k_t+1}$. If $t \in supp_p(g)$, let $W_t = W^t_{k_t} \setminus \overline{W^t_{k_t+2}}$; otherwise let $W_t = X_t$. By regularity, for every $t \in T$, we have that $g(t) \in W^t_{k_t} \setminus W^t_{k_t+1} \subseteq W_t$ and W_t is open. Then,

$$W = (\square_{t\in T} W_t) \cap (\square_{t\in T} W^t_{e_\alpha(t)+2})$$

is a neighborhood of g.

Now, suppose for a contradiction there is $f \in W \cap G$. There is $\beta < \mathfrak{d}(\kappa)$ such that $d_\beta = \Phi_p(f)$. Observe that $supp_p(g) = supp_{\vec{\omega}}(d_\alpha)$ and for all $t \in supp_p(g)$ $(d_\alpha(t) = k_t)$.

- If $\alpha < \beta$, then the property above gives us $t \in supp_{\vec{\omega}}(d_\alpha)$ such that $d_\alpha(t) + 2 < d_\beta(t)$. That is, $f(t) \notin W^t_{k_t} \setminus \overline{W^t_{k_t+2}} = W_t$, this contradicts $f \in W$.
- If $\beta < \alpha$, then we had that $N_{e_\alpha} \cap D_\alpha = \emptyset$. Since d_β is an element of D_α, exists $t \in supp_{\vec{\omega}}(d_\beta)$ such that $e_\alpha(t) > d_\beta(t)$. Thus, $d_\beta(t) < e_\alpha(t) + 2$. It follows that $f(t) \notin W^t_{e_\alpha(t)+2}$ and also contradicts that $f \in W$.

This concludes the proof. ■

By Theorem 11, box products of regular first countable spaces are discretely generated. However, it is still unknown the result if we weaken the property for Fréchet–Urysohn or sequential spaces. If box products of regular first countable spaces are discretely generated, are the box products of Fréchet–Urysohn or sequential spaces discretely generated?

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